Math 38 — Graph Theory
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Bounds on the chromatic number
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Last class, I introduced proper colorings of graphs, and the chromatic number. We also looked at some bounds on the chromatic number, and we keep exploring bounds on the chromatic number today.

So far, we know:

- The chromatic number can be bounded in terms of the independence number and the clique number: $\chi(G)\geq\omega(G)$ and $\chi(G)\leq|V|/\alpha(G)$.
- The chromatic number can be bounded in terms of the maximum degree: $\chi(G)\leq\Delta(G)+1$.

These bounds are easy to check, but they are not the best possible.

Another upper bound

Theorem (Brooks, 1941)
If G is connected, and is not the complete graph nor an odd cycle, $\chi(G)\leq\Delta(G)$.

Examples and special cases

If $\Delta(G)=0$, then G has 1 vertex (because it is connected), and is thus the complete graph. So no graph in this case satisfies the hypotheses of the theorem.

If $\Delta(G)=1$, then G has 2 vertices, and this is again the complete graph.

If $\Delta(G)=2$, G is either a cycle or a path. Open paths and even cycles are bipartite, so their chromatic number is 2, which also is the maximum degree. Even cycles are excluded from the hypothesis of the theorem.

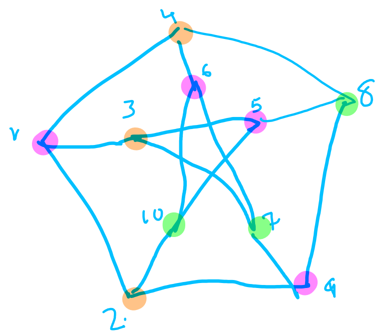
Complete graphs don't satisfy the inequality, as their chromatic number is one more than the maximum degree (every vertex must have different colors).

The hypothesis that the graph is connected is needed to avoid the case of having only isolated vertices.

Not complete, maximum degree is 0.
Chromatic number is 1.

Notice that, whenever a graph with n vertices is not the complete graph, the chromatic number is at most $n-1$: Since there is at least one pair of non-adjacent vertices in a non-complete graph, they can be the same colors. So n colors are never needed if the graph is not complete.

Example: Coloring the Petersen graph using the greedy algorithm



The Petersen graph is 3-regular.
It satisfies the hypothesis of the theorem, so it must have maximum degree 3. That means there exists an ordering of the vertices that allows it.

Proof of Brooks' Theorem

We already inspected the case where the largest degree is at most 2, so assume $\Delta(G)=k$ is at least 3.

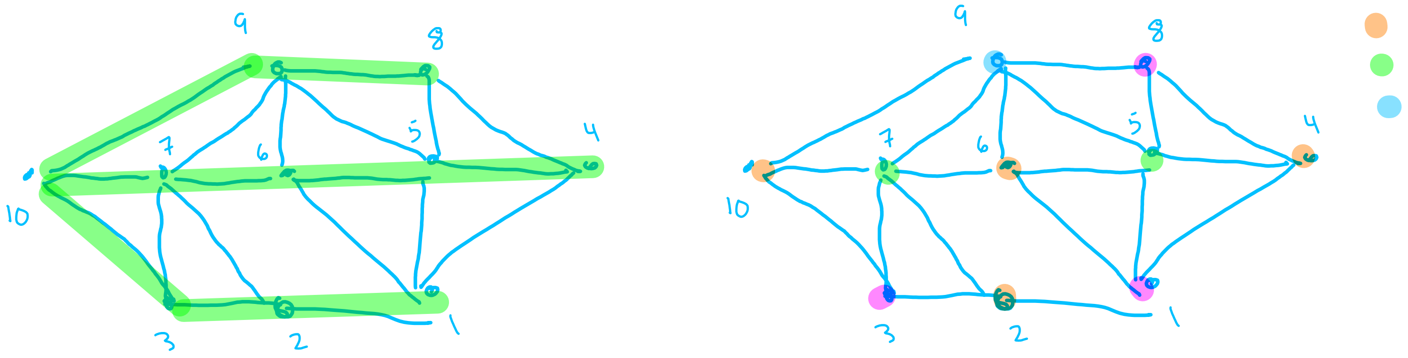
If G is not k-regular:

Then, there is a vertex v with degree less than k. Let T be a spanning tree in G (which is possible since the graph is connected). We will use this spanning tree for ordering the vertices. The goal is to find the right ordering for the vertices, and then apply the greedy algorithm from last lecture.

- Number vertex v with n (last vertex to be colored).
- Label the other vertices in decreasing order on paths leaving v in T.
- Color the vertices using the greedy algorithm from last lecture.

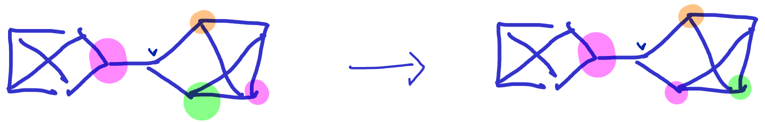
Every time we color a new vertex u (that is not v), there are at most $k-1$ of its neighbors that have been previously colored, so k colors are enough.

For the last step, we know that v has at most k−1 neighbors, so in the worst case, a k-th color will be necessary to color it.
In total, k colors are enough if the graph is not k-regular.

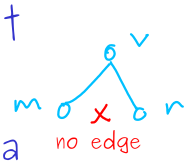


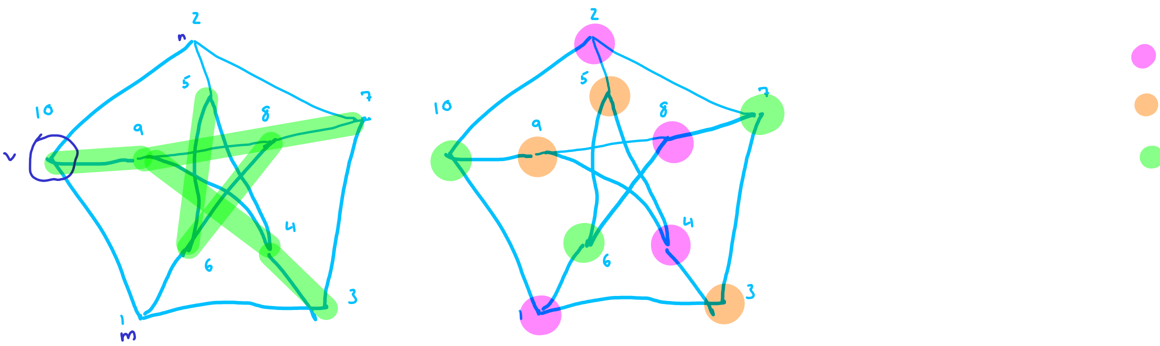
A similar process holds if the graph is k-regular, but there are two cases:

- There is a cut-vertex v. Then, G−{v} is disconnected, and each component can be colored with k colors. Place the colors in the components so that every vertex incident to v has the same color. Then, v can be colored using any other color, so G is k-colorable.



- There is no cut-vertex, meaning that G is 2-connected.
  If G has a vertex v with two neighbors that are not adjacent m and n such that G−{m,n} is connected, we can use a similar argument. We label m and n by 1 and 2, and create a spanning tree in G−{m,n}. Starting from v, we label the vertices in decreasing order and obtain a proper k-coloring of G because the last vertex has two vertices (m and n) colored the same.





I claim there is always such a triple of vertices when G is 2-connected and k-regular, with $k \geq 3$. (The details of this are in the textbook.)

Subgraph, cliques and chromatic number

Proposition

If H is a subgraph of G, $\chi(H) \leq \chi(G)$.

Proof

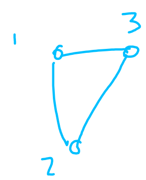
All the edges of H are in G, so the vertices of H cannot be colored with fewer than $\chi(H)$ vertices (however, if we add edges, they might need more colors).

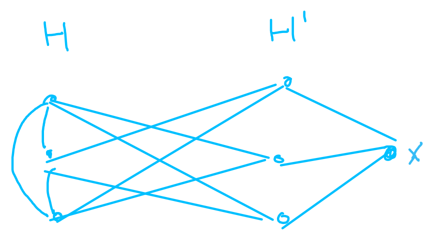
This is similar to the proposition we had in last lecture: $\chi(G) \geq \omega(G)$. However, cliques are not needed to have large chromatic number.

Example: Mycielski's construction

From a simple graph G, construct a graph G' in the following way: Let H and H' be two copies of G, but delete all edges from H'. If vertices u and v are adjacent in G, draw an edge between u in H and v' in H' (the copy of v in H'). Add an extra vertex x and connect it to all the vertices in H'.

G





G'

Notice that u and u' are never adjacent.

If G has chromatic number k, then G' has chromatic number $k+1$: The colors in H and in H' can be the same. In G, u and v can have the same color if they are not adjacent. Hence, u and v' (as well as v and u') are not adjacent in G', so they can have the same color. Hence x is the only vertex with a new color added.

So graphs obtained by iterating this process can have arbitraty large chromatic number.

Question: What is the clique number of a graph obtained with the Mycielski's contruction?

Proposition

Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges.

Proof

Consider an optimal coloring of the graph. Since it uses the minimum number of colors, there is at least one edge connecting two color classes; otherwise, there are two classes (blue and red) with no edges between the two classes, and all the red vertices can be colored blue. Hence, we need at least one edge per pair of colors, that is $\binom{k}{2}$ edges.

This is achieved by complete graphs K_k, plus $n-k$ isolated vertices. ☒

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Sections 5.1 and 5.2