

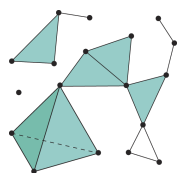
The goal of today's lecture is to give a shortcut for computing the chromatic polynomial of many graphs.

Geometry

Simplex (triangle, tetrahedron, etc.)

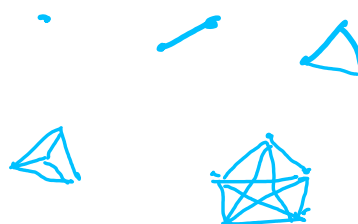


Simplicial complex
(gluing simplices together)
Every face of a simplicial complex is a simplex.



Graph theory

Clique



Chordal graph, i.e.
graph that admit a simplicial
elimination ordering.

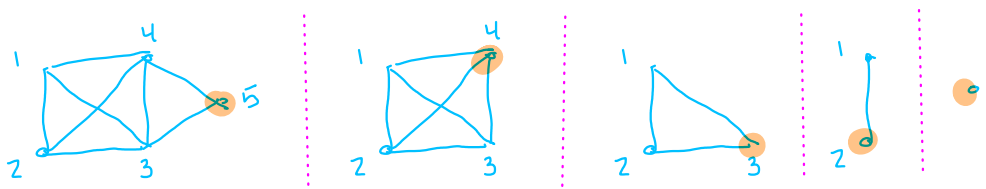


A vertex is simplicial if its neighbors form a clique.



- simplicial
- Not simplicial

A simplicial elimination ordering is an ordering of the vertices v_n, \dots, v_1 such that the vertex v_i is simplicial in $G - \{v_n, \dots, v_{i+1}\}$.



The ordering above is a simplicial elimination ordering.

Example

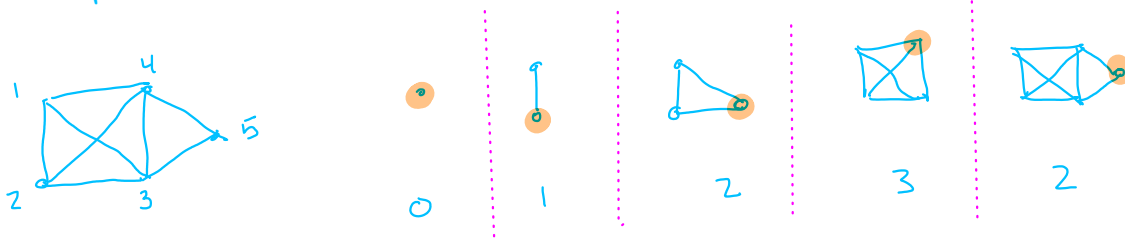
- For complete graphs, any ordering is a simplicial elimination ordering.
- For trees, an ordering starting at the leaves and going inwards is a simplicial elimination ordering.
- The cycle of length 4 has no simplicial ordering.



Proposition

The chromatic polynomial of a graph with simplicial ordering v_1, \dots, v_n is the product of the $k - d^*(v_i)$'s, where $d^*(v_i)$ is the degree of v_i in the induced subgraph with vertices $\{v_1, \dots, v_i\}$.

Example



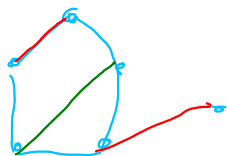
$$\chi(G; k) = k(k-1)(k-2)^2(k-3)$$

Sketch of proof

When it is added, v_i has $d^*(v_i)$ options for coloring it. Doing this process, we count all the options for coloring the graph with k colors.

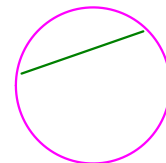
Chordal graphs

A chord of a cycle C is an edge not in C that whose endpoints are in C .



Chord

Not a chord



A cycle is chordless if it has length at least 4 and no chord.

A graph is chordal if it is simple and it has no chordless cycle (as induced subgraphs).

Theorem (Dirac, 1961)

A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.

Lemma (Voloshin, 1982, or Farber–Jamison, 1986)

Every chordal graph has a simplicial vertex.

The proof of the lemma is omitted. It is in the textbook as Lemma 5.3.16.

Proof of the theorem

Necessity: If it is not chordal, there is a chordless cycle C . In C , that has length at least 4, none of the vertices is simplicial, and no vertex can be the first one to be picked in C for the ordering.

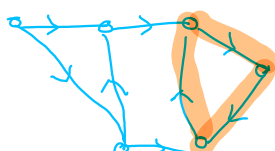
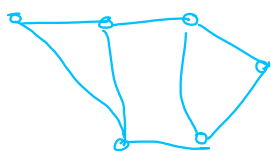
Sufficiency: Using the lemma, we know that every chordal graph G has a simplicial vertex v . Delete v . Then $G - \{v\}$ is chordal, so we can apply the lemma again, creating an ordering.



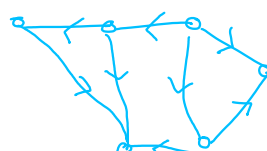
Acyclic orientations

What is the meaning of $\chi(G; -1)$?

An orientation of a graph G is a digraph that has G as an underlying graph.



orientation



acyclic orientation

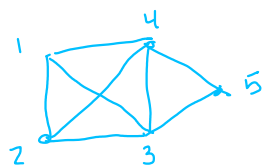
An orientation is acyclic if it has no cycle.

Theorem (Stanley, 1972)

The absolute value of $\chi(G; -1)$ is the number of acyclic orientations of G .

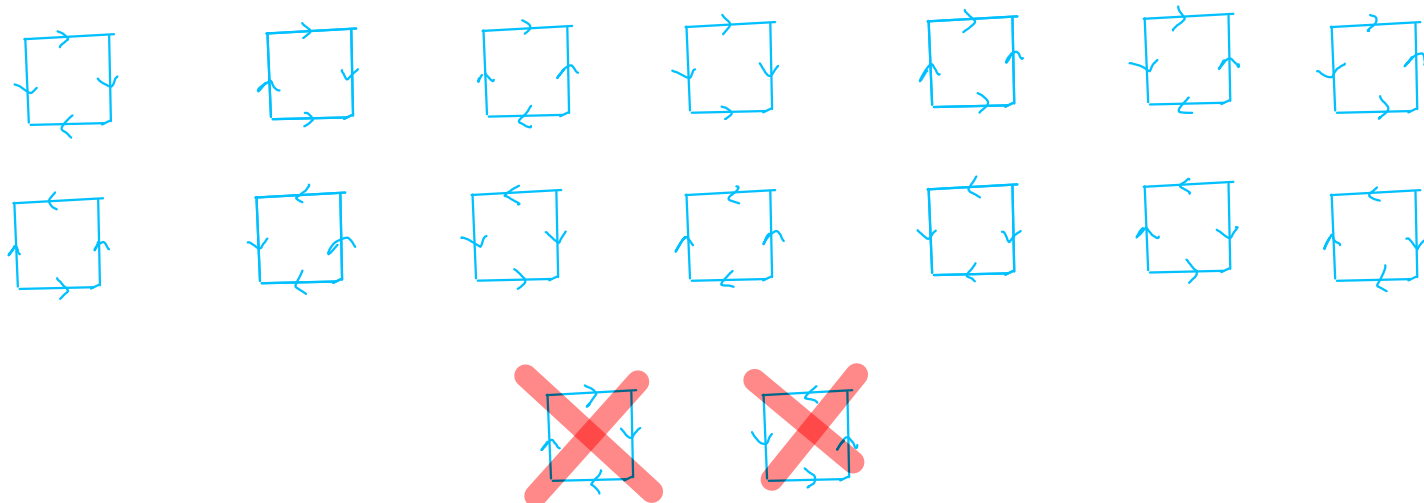
Homework Read proof of Theorem 5.3.27 in the textbook.

Examples



has chromatic polynomial $\chi(G; k) = k(k-1)(k-2)^2(k-3)$, so it has 72 acyclic orientations.

C_4 has chromatic polynomial $\chi(C_4; k) = k(k-1)(k^2 - 3k + 3)$, and $\chi(C_4; -1)$ is 14.



Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 5.3.