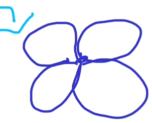
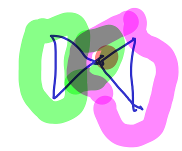
6.2.5 Lemma : Every minimal nonplanar graph is 2-connected.

We will prove that it is connected (connectivity is not 0) and it does not have a cut-vertex. That is equivalent to being 2-connected.

It is connected: If a nonplanar graph has more than one component, there is at least one component that is nonplanar. Hence, if a nonplanar graph is disconnected, it cannot be minimal, as one of its component is nonplanar.

Lobes are spread out so they don't overlap.





There is no cut-vertex.

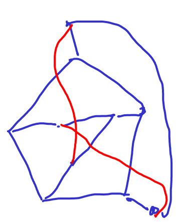
If v is a cut-vertex of G, then a lobe of G is the subgraph induced by one component of G−v and v.

If there is more than one lobe, the graph cannot be a minimal nonplanar graph. The reason for this is that a lobe would have to be nonplanar, which would contradicts the minimality of G as a nonplanar graph.

If a graph has only one lobe with respect to the vertex v, it means that deleting v creates only one component. So v is not a cut-vertex.

Conclusion : If the graph is a minimal nonplanar graph, it is connected and has no cut-vertex, so it is 2-connected.

Exercise 6.2.5 What is the minimum number of edges that must be deleted from the Petersen graph to obtain a planar subgraph?



In blue, a plane subgraph with 13 edges.

Recall (from the first week) that the shorter cycle in the Petersen graph has length 5. A face always contains a cycle on its boundary. So the length of a face in the Petersen graph is at least 5.

Recall also that twice the number of edges is the sum of the length of the faces.

$$2\#E = \sum_{\text{faces} f_i} l(f_i) \geq \sum_{\text{faces} f_i} 5 = 5 \ \#\text{faces}$$

Along with Euler's formula, we know that n−e+f=2, which implies the following

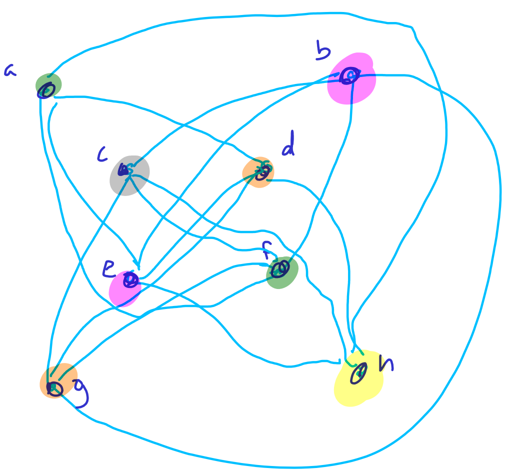
$$n - e + f = 2 \implies 5n - 5e + 5f = 10 \implies 10 \leq 5n - 5e + 2e = 5n - 3e$$

Since the number of vertices is fixed and is 10, then this is

$$10 \leq 50 - 3e \implies e \leq \frac{40}{3}$$

Hence, the number of edges in a planar subgraph of the Petersen graph is at most 13. The minimum number of edges to be deleted is 2.

Exercise 6.2.1 The complement of the hypercube H_3 is not planar.



We'll look for a forbidden minor and use Wagner's Theorem.

By contracting be, af and cg, we find K5 as a minor.