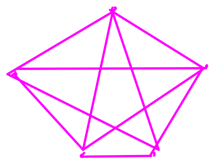
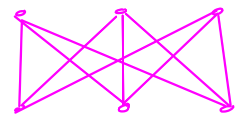
Math 38 — Graph Theory   Nadia Lafrenière

Characterization of planar graphs

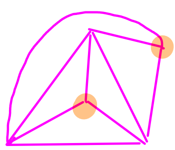
06/01/2020

Recall from last class that K_5 and $K_{3,3}$ are not planar:

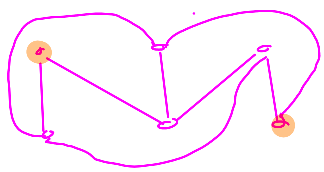




They are minimal nonplanar subgraphs, as removing any edge from them makes them planar:



$K_5 - e$



$K_{3,3} - e$

The goal of today's lecture is to prove that every nonplanar graph "contains" (not necessarily as subgraph) either K_5 or $K_{3,3}$.

Proposition

(A) If G is a nonplanar graph, adding vertices or edges to it will never make it planar.

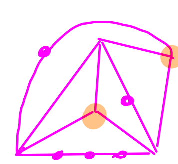
(B) Subdividing an edge (i.e. changing an edge for a path), does not change planarity.

Proof

For (A), we will never be able to draw in a planar way the vertices and edges that are already in G.

For (B), notice that subdividing is like adding a vertex in the middle of an edge. If the original graph is nonplanar, we use (A) to prove it is not planar.

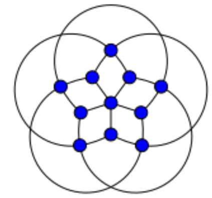
If it is originally planar, we can add vertices in the middle of an edge and still be able to draw it.

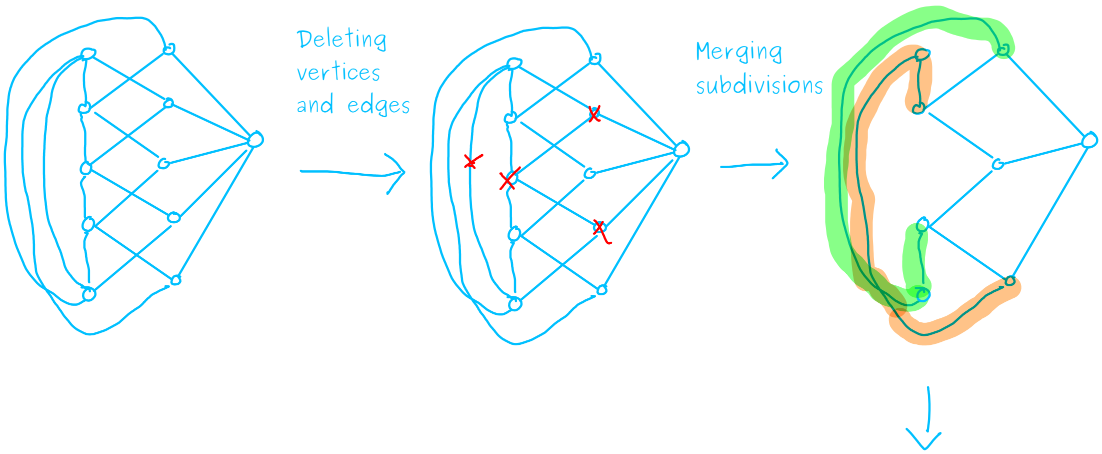


Corollary

If G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, it is not planar.

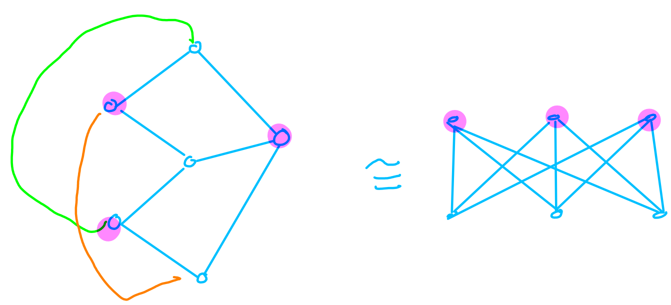
Example: The Grötzsch Graph (M_4) is not planar:





A graph that is a subdivision of K_5 or $K_{3,3}$ is a Kuratowski graph.

G has a Kuratowski subgraph if it has a subgraph that is a Kuratowski graph.



Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not have a Kuratowski subgraph.

Notice that the left-to-right direction of Kuratowski's Theorem is the corollary above. The right to left part should be more surprising.

This theorem means that there is a way of knowing whenever a graph is planar, without necessarily drawing it, or proving it will never be possible.

(Very coarse) sketch of proof. Details are in the textbook.

- We are looking for nonplanar subgraphs. We can focus on minimal nonplanar subgraphs (as the whole graph will not be planar if they exist). The claim we wish to prove is that minimal nonplanar subgraphs are Kuratowski graphs.
- We prove that a minimal nonplanar graph is 2-connected, and that it would be 3-connected if it had no Kuratowski subgraph.
- Tutte's theorem says that a 3-connected subgraph with no Kuratowski subgraph is planar (and can be drawn using only convex polygons).
- We conclude that there is no minimal nonplanar subgraph that does not contain a subdivision of K_5 or $K_{3,3}$, proving Kuratowski's Theorem.

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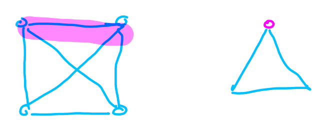
Wagner's Theorem

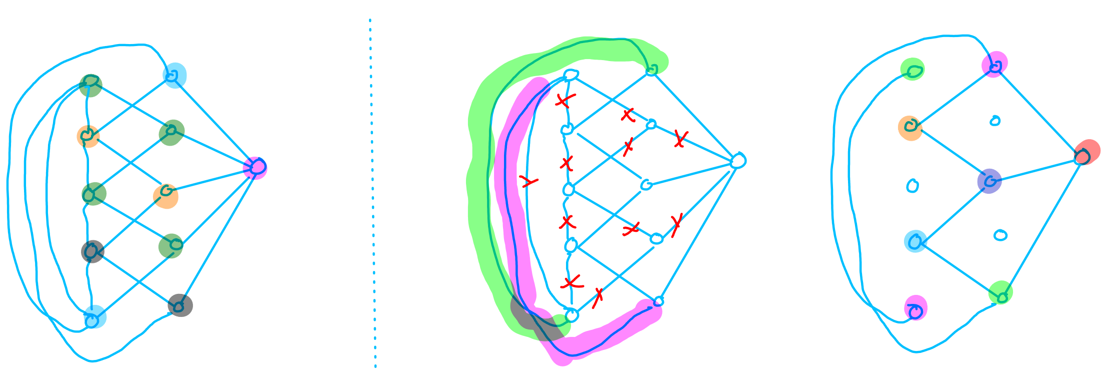
Just as spanning trees and chromatic polynomials, planarity has its own version of a deletion-contraction theorem. In many cases, Wagner's Theorem is much easier to use than Kuratowski's Theorem.

A graph H is a <u>minor</u> of a graph G if H can be obtained from G by contracting and deleting edges, and deleting vertices.



K_3 is a minor of K_4

Example

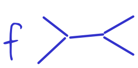


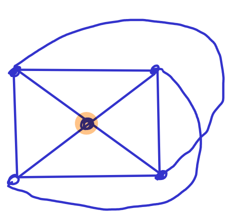
The Grötzsch graph has both forbidden minors: K_5 and $K_{3,3}$

Theorem (Wagner, 1937)
A graph is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of it.

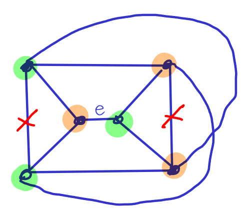
Lemma
If G has no Kuratowski subgraph, then $G \cdot e$ has no Kuratowski subgraph, for any edge e.

Proof (of Lemma)
We prove the contrapositive: We fix an edge e, and we prove that if $G \cdot e$ has a Kuratowski subgraph, then so does G. WLOG, we can assume G is simple, because multiple edges and loops don't change anything to planarity.

Let H be a Kuratowski subgraph in $G \cdot e$.
Consider the following cases:
1- e has one endpoint of degree at most 2 in H: Then H is a subdivision of $H \cdot e$, and H is a Kuratowski subgraph.
2- $e = xy$ has both endpoints with degree at least 3 in H: the vertex, z, that is obtained by merging x and y also has degree at least 4.
Since H is either a subdivision of K_5 or $K_{3,3}$, if it has a vertex of  degree at least 4, it is a subdivision of K_5. In fact, this is a vertex of degree exactly 4, and has been obtained by merging 2 vertices of degree 3.



$H \cdot e \cong K_5$



H contains $K_{3,3}$

Hence, in both cases, H contains a Kuratowski subgraph, and so does G (that contains H).

That lemma is the key to prove Wagner's Theorem from Kuratowski's Theorem.

Proof of Wagner's Theorem

=> (If a graph is planar, then it does not contain K_5 nor $K_{3,3}$ as a minor.)

Using the lemma above, we know that contraction of edges preserves planarity, and so does deleting edges or vertices from planar graphs. Hence, all minors of a planar graph must be planar, so K_5 and $K_{3,3}$ are not minors.

<= (Contrapositive: If a graph is not planar, it contains K_5 or $K_{3,3}$ as minors.)

Using Kuratowski's Theorem, a nonplanar graph contains a subgraph H that is a subdivision of either K_5 or $K_{3,3}$. Delete all the edges and vertices that do not belong to H. Replace all vertices of degree 2 and their 2 incident vertices by one edge. At the end of the process, you get K_5 or $K_{3,3}$, that was obtained from H by contraction and deletion. So K_5 or $K_{3,3}$ is a minor of G.

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Remarks

- From a subdivision, it is easy to get a minor. The converse (getting a subdivision from a minor) is not that easy, and not even always possible.
- Wagner's Theorem is often much handier than Kuratowski's Theorem.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001.
Section 1.1