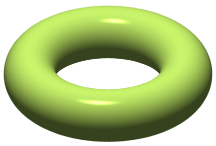
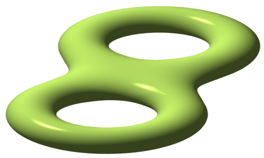
Coloring on surfaces, in general

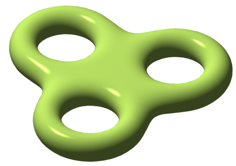
The genus of an orientable surface is its number of handles.



Genus 1



Genus 2



Genus 3

The plane and the sphere have genus 0

A graph is embeddable on a surface if it can be drawn on it without edges crossing.
A planar graph is embeddable on a plane.

Theorem
An orientable surface with genus g has Euler's characteristic $2-2g$.
That implies a graph G with n vertices and e edges that is embedddable on a surface of genus g satisfies $n-e+f=2-2g$, where f is the number of faces.

Seven color Theorem

Theorem
If G is embeddable on a torus (genus 1), then G is 7-colorable.

Proof
We know, by Euler's formula for the torus, that $n-e+f=0$. We also know that $2e=$(sum of the length of the faces)$\geq 3f$. This means that $3n-3e+3f=0$ (or $3n-3e=-3f$, so $3n-3e\geq -2e$, and $3n\geq e$).
Thus, there needs to be a vertex with degree at most 6 for every graph that is embeddable on a torus.
This implies that every subgraph of G has a vertex of degree at most 6.

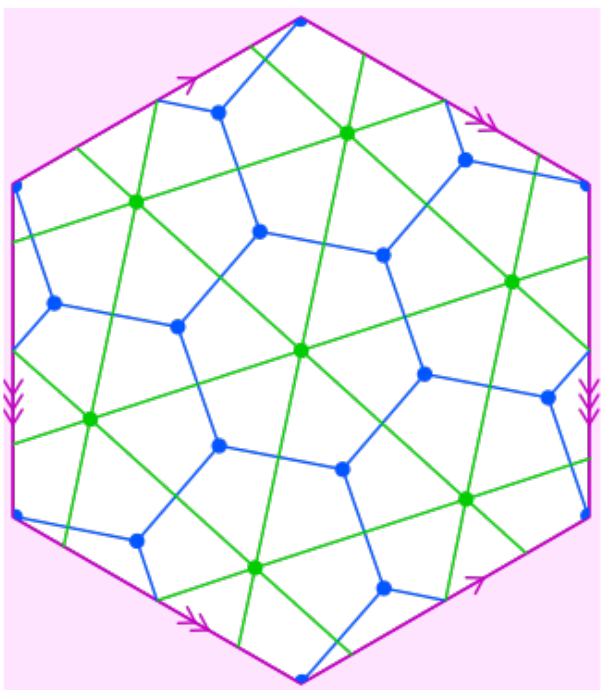
Using the lemma from the lecture notes (in the section for the Six Color Theorem), that means that G is 7-colorable. ◪

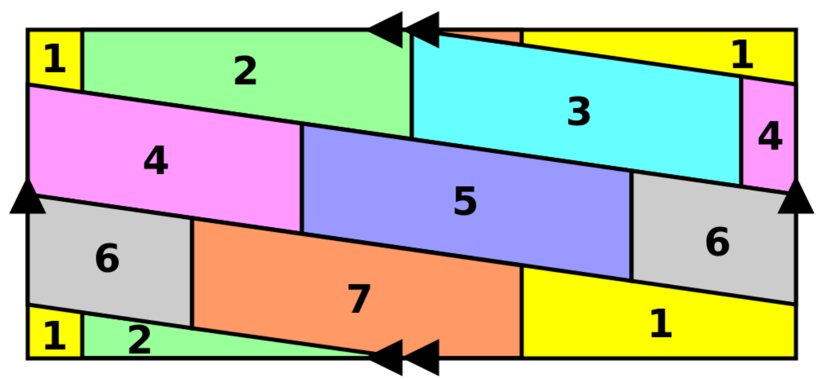
Theorem

There exist some graphs that are embeddable on the torus that have chromatic number 7.

Proof

K_7 can be embedded on the torus, as shown below. As every complete graph, its chromatic number is its number of vertices.





Source of the picture:
Maproom, on Wikipedia