The number of numerical semigroups of a given genus

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The coin problem

Given coins of denominations $c_1, c_2, \ldots, c_m$,

- what is the largest amount that cannot be obtained? (Frobenius problem)
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$$0, 3, 5, 6, 8, 9, 10, \ldots$$

Such a set is called a *numerical semigroup*.

Definitions

A *numerical semigroup* is a set $\Lambda \subseteq \mathbb{N}_0 = \{0, 1, 2, \dots\}$ satisfying:

- $0 \in \Lambda$,
- Λ is closed under addition,
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Example:

$\Lambda = \{0, 4, 6, 8, 9, 10, 11, \dots\}$ $\qquad f = 7,\ g = 5$

n_g

Let n_g be the number of numerical semigroups of genus g.

$g = 1$: $\{0, 2, 3, 4, \dots\}$

$g = 2$: $\{0, 2, 4, 5, 6, \dots\}$ $\{0, 3, 4, 5, 6, \dots\}$

$g = 3$: $\{0, 2, 4, 6, 7, 8, \dots\}$ $\{0, 3, 4, 6, 7, 8, \dots\}$
$\{0, 3, 5, 6, 7, 8, \dots\}$ $\{0, 4, 5, 6, 7, 8, \dots\}$

g	1	2	3	4	5	6	7	8	9	10	11	12	13
n_g	1	2	4	7	12	23	39	67	118	204	343	592	1001

Minimal generators

Every numerical semigroup Λ with $g \geq 1$ has a unique minimal set of generators $\mu_1, \mu_2, \ldots, \mu_m$.

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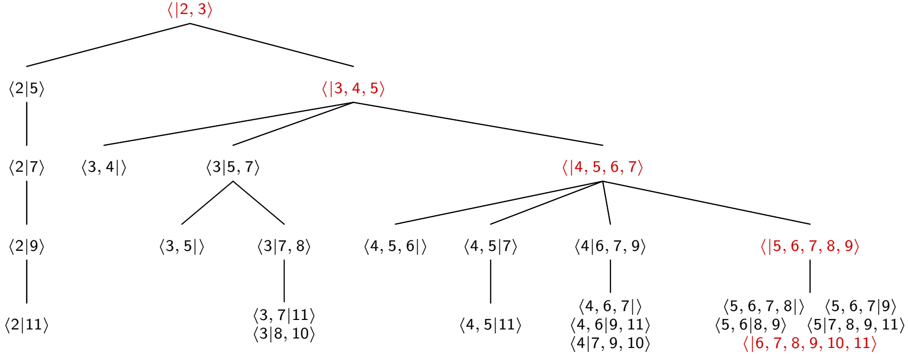
Example: $\{0, 4, 6, 8, 9, 10, 11, \dots\} = \langle 4, 6 | 9, 11 \rangle$,
$\{0, 4, 6, 7, 8, 9, 10, 11, \dots\} = \langle 4 | 6, 7, 9 \rangle$

The tree $\mathcal{T}$ of numerical semigroups



Consider the tree $\mathcal{T}$ with root $\{0, 2, 3, 4, \dots\} = \langle | 2, 3 \rangle$ where

- the parent of each Λ is $\Lambda \cup \{f\}$,
- the children of each $\Lambda = \langle \mu_1, \dots, \mu_r | \mu_{r+1}, \dots, \mu_{r+e} \rangle$ are $\Lambda \setminus \{\mu_{r+i}\}$, with $1 \leq i \leq e$.

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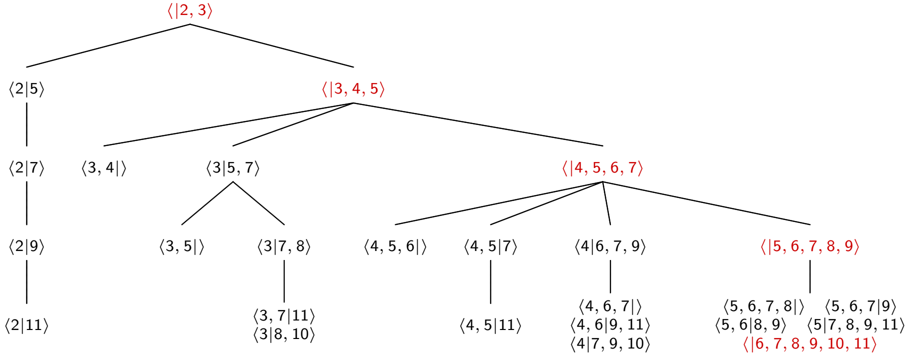

The number of nodes at level g is n_g. We will bound n_g by approximating this tree with simpler trees, keeping track of the number of effective generators of each node.

Ordinary semigroups

$O_g = \{0, g+1, g+2, g+3, \dots\} = \langle | g+1, g+2, \dots, 2g+1 \rangle$
is the *ordinary semigroup* of genus g.

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or equivalently as

$$\overline{(e)} \longrightarrow (0)(1)\dots(e-3)(e-1)\overline{(e+1)}.$$

Non-ordinary semigroups

Let $\Lambda = \langle \mu_1, \dots, \mu_r | \mu_{r+1}, \dots, \mu_{r+e} \rangle$ be a non-ordinary semigroup. Then, for $1 \leq i \leq e$,

$$\Lambda \setminus \{\mu_{r+i}\} = \begin{cases} \langle \mu_1, \dots, \mu_{r+i-1} | \underbrace{\mu_{r+i+1}, \dots, \mu_{r+e}}_{e-i \text{ effective gen.}} \rangle & \text{or} \\ \langle \mu_1, \dots, \mu_{r+i-1} | \underbrace{\mu_{r+i+1}, \dots, \mu_{r+e}, \mu_1 + \mu_{r+i}}_{e-i+1 \text{ effective gen.}} \rangle. \end{cases}$$

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$$(3) \longrightarrow (0)(2)(3)$$

In general, $(e) \longrightarrow (j_1)(j_2)\dots(j_e)$, where $j_i \in \{i-1, i\}$.

A lower bound

Consider the generating tree with root $\overline{(2)}$ and succession rules

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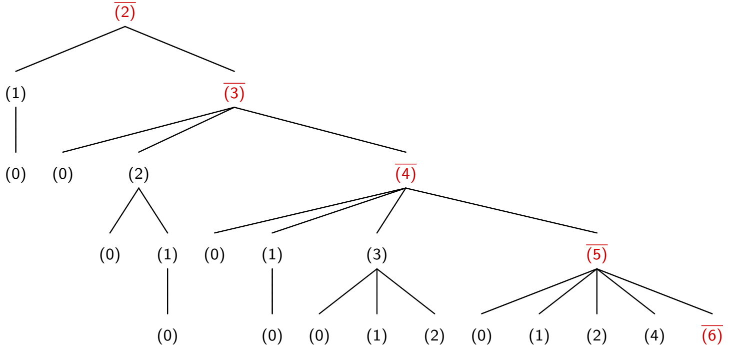

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From the succession rules, the generating function for the number of nodes at each level is

$$\frac{t(1+t+t^2)}{1-t-t^2} = t + 2t^2 + 4t^3 + 6t^4 + 10t^5 + \cdots = t + \sum_{g\geq 2} 2F_g t^g.$$

So, for $g \geq 2$,

$$n_g \geq 2F_g.$$

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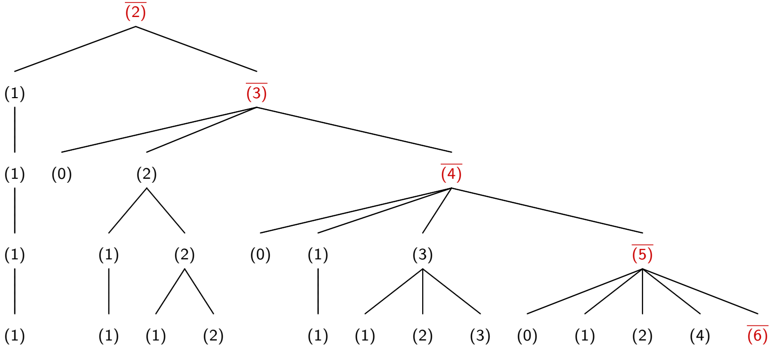

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From the succession rules, the generating function for the number of nodes at each level is

$$\frac{t(1-t-t^3)}{(1-t)(1-2t)} = t + 2t^2 + 4t^3 + 7t^4 + 13t^5 + \ldots$$

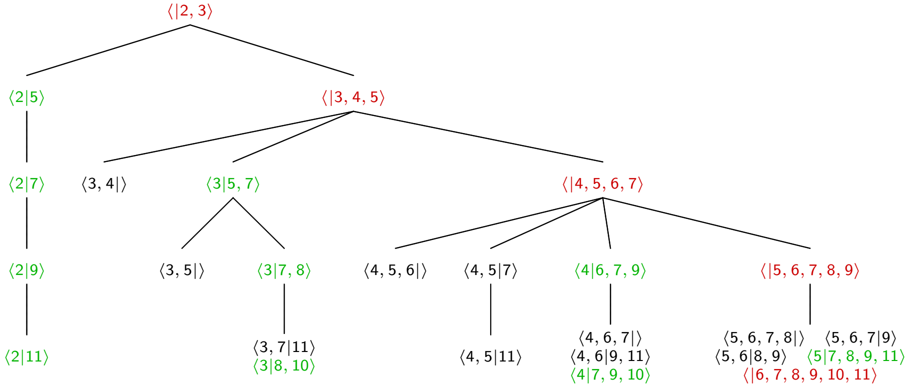
So, for $g \geq 3$,

$$n_g \leq 1 + 3 \cdot 2^{g-3}.$$

A better lower bound

Consider the semigroups with only one generator less than f:

$$P_{g,i} = \langle g+1 | g+i, g+i+1, \ldots, \widehat{d(g+1)}, \ldots, 2g+i \rangle.$$



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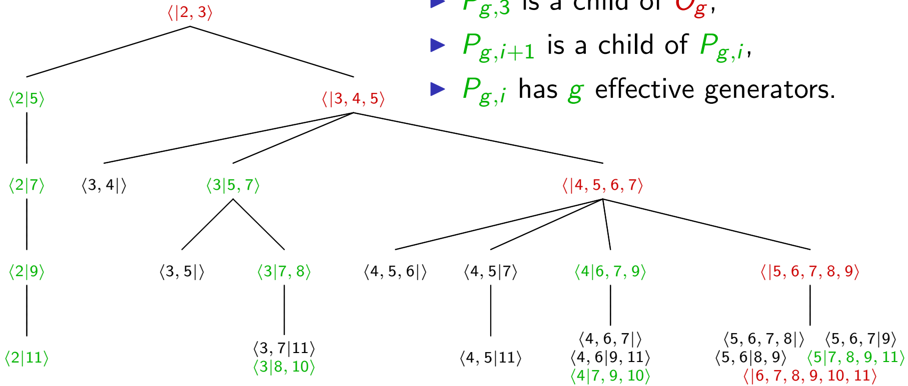

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The succession rules for the new tree are

$$\overline{(e)} \longrightarrow (0)(1)\ldots(e-3)\widetilde{(e-1)}\overline{(e+1)},$$
$$\widetilde{(e)} \longrightarrow (0)(1)\ldots(e-2)\widetilde{(e)},$$
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Counting the nodes gives an improved lower bound:

$n_g \geq F_{g+2} - 1 \geq 2F_g$.

An even better lower bound

Idea: Use a second label to keep track of the number of strong generators of each semigroup. An effective gen. $\mu \in \Lambda$ is called *strong* if $\mu + \mu_1$ is a generator of $\Lambda \setminus \{\mu\}$.

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We bound the number of strong gen. in terms on the number of strong gen. of the parent. The succession rules become

$$\begin{aligned} \overline{(e)} &\longrightarrow (0,0)(1,0)\dots(e-3,0)\widetilde{(e-1)}_2\overline{(e+1)}, \\ \widetilde{(e)}_k &\longrightarrow (0,0)(1,0)\dots(e-\sigma-1,0)(e-\sigma+1,0)(e-\sigma+2,1)\dots(e-1,\sigma-2)\widetilde{(e)}_{k+1}, \\ (e,s) &\longrightarrow (0,0)(1,0)\dots(e-s-1,0)(e-s+1,0)(e-s+2,1)\dots(e,s-1). \end{aligned}$$

where

$$\sigma = \sigma(e,k) := \begin{cases} k & \text{if } 2 \le k \le \lceil e/2 \rceil, \\ k-1 & \text{if } \lceil e/2 \rceil < k \le e, \\ e & \text{if } k > e. \end{cases} \qquad (\#\text{ of strong gen. of } P_{e,k+1})$$

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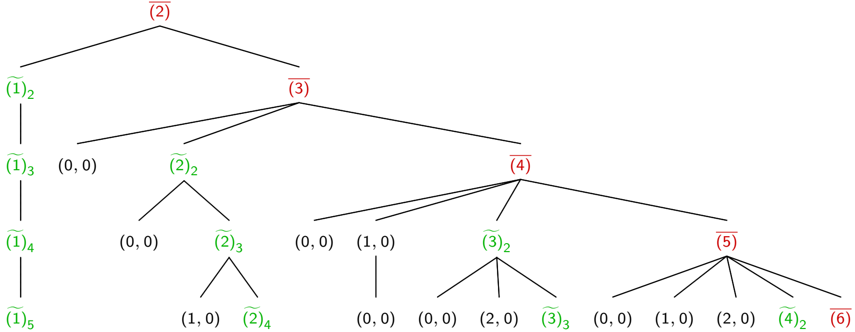

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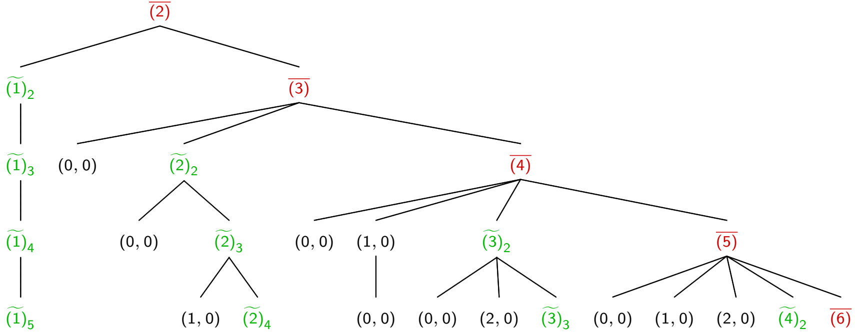

The coefficients of its corresponding generating function

$$\frac{t\,(1-t^2-2t^3-3t^4+t^5+2t^6+3t^7+3t^8+t^9)}{(1+t)(1-t)(1-t-t^2)(1-t-t^3)(1-t^3-2t^4-2t^5-t^6)}$$

give a better lower bound on n_g.

A better upper bound

Idea: use a second label to keep track of the number of healthy generators of each semigroup. An effective gen. $\mu \in \Lambda$ is called *healthy* if $\mu + \mu_1 \leq 2g + 3$. Strong generators are always healthy.

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We bound the number of healthy gen. in terms on the number of effective and healthy gen. of the parent. The succession rules become

$$\begin{aligned} \overline{(e)} &\longrightarrow (0,0)(1,0)\dots(e{-}4,0)(e{-}3,\min\{1,e{-}3\})(e{-}1,\min\{2,e{-}1\})\overline{(e{+}1)}, \\ (e,h) &\longrightarrow (0,0)(1,0)\dots(e{-}h{-}2,0)(e{-}h{-}1,\min\{1,e{-}h{-}1\}) \\ &\quad\; (e{-}h{+}1,\min\{2,e{-}h{+}1\})(e{-}h{+}2,\min\{3,e{-}h{+}2\})\dots(e,\min\{h{+}1,e\}). \end{aligned}$$

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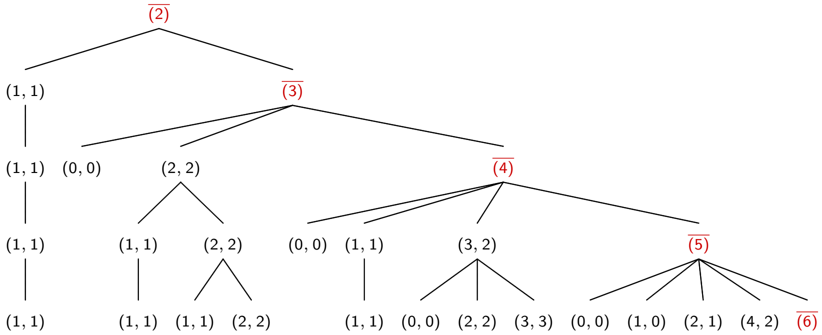

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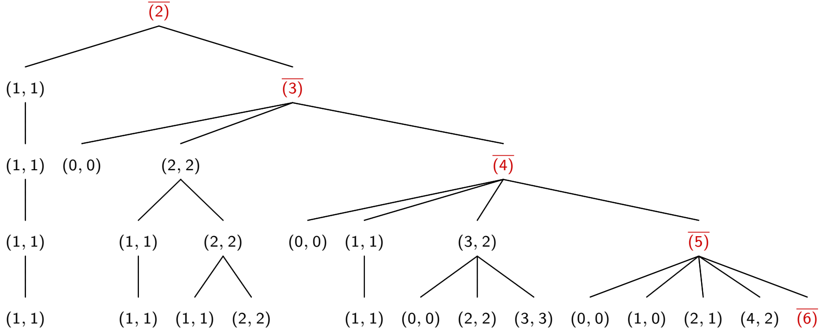

The coefficients of its corresponding generating function

$$t\,\frac{2-3t+t^2-4t^3+3t^4-2t^5+t(1-t-t^3)\sqrt{(1+2t)/(1-2t)}}{2(1-3t+3t^2-3t^3+4t^4-3t^5+2t^6)}$$

give the best known upper bound on n_g.

g	$2F_g$	$F_{g+2}-1$	lower bound	$\mathbf{n_g}$	upper bound	$1+3\cdot 2^{g-3}$
1		1	1	1	1	
2	2	2	2	2	2	
3	4	4	4	4	4	4
4	6	7	7	7	7	7
5	10	12	12	12	13	13
6	16	20	22	23	24	25
7	26	33	37	39	44	49
8	42	54	62	67	81	97
9	68	88	104	118	151	193
10	110	143	175	204	280	385
11	178	232	291	343	525	769
12	288	376	482	592	984	1537
13	466	609	796	1001	1859	3073
14	754	986	1315	1693	3511	6145
15	1220	1596	2166	2857	6682	12289
16	1974	2583	3559	4806	12709	24577
17	3194	4180	5838	8045	24334	49153
18	5168	6764	9569	13467	46565	98305
19	8362	10945	15665	22464	89626	196609
20	13530	17710	25612	37396	172381	393217
21	21892	28656	41831	62194	333262	786433
22	35422	46367	68270	103246	643733	1572865
23	57314	75024	111337	170963	1249147	3145729
24	92736	121392	181438	282828	2421592	6291457
25	150050	196417	295480	467224	4713715	12582913
26	242786	317810	480938	770832	9165792	25165825
27	392836	514228	782408	1270267	17888456	50331649
28	635622	832039	1272250	2091030	34873456	100663297
29	1028458	1346268	2067870	3437839	68212220	201326593
30	1664080	2178308	3359757	5646773	133269997	402653185
31	2692538	3524577	5456862	9266788	261167821	805306369
32	4356618	5702886	8860132	15195070	511211652	1610612737

Open problems

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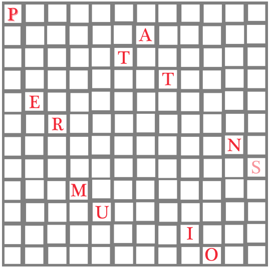
- $n_{g+1} \geq n_g$ for all g,
- $n_{g+2} \geq n_g + n_{g+1}$ for all g.



Eighth International Conference on **Permutation Patterns**, *PP 2010*

August 9-13, Dartmouth College, Hanover, NH

Invited speakers:

- Nik Ruškuc, University of St Andrews
- Richard Stanley, MIT

http://math.dartmouth.edu/~pp2010