Partial rank symmetry of distributive lattices for fences

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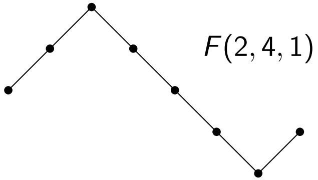
Fence posets

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ with $\beta_i \geq 1$ for all i.

Definition

The *fence* $F(\beta)$ is the poset consisting of chains of lengths $\beta_1, \beta_2, \ldots, \beta_s$, where the ith and $(i+1)$st chains share their maximum element if i is odd, and they share their minimum element if i is even.



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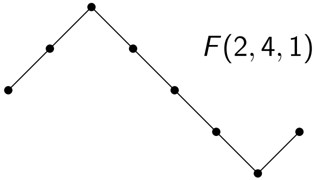

$F(2,4,1)$

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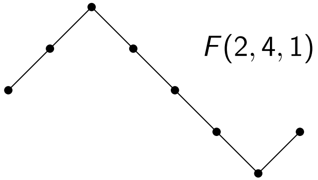

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Let $n = |F(\beta)| = \beta_1 + \cdots + \beta_s + 1$.

Lower order ideals

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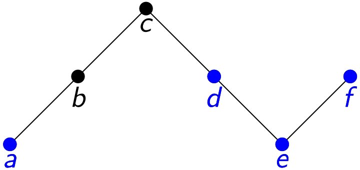

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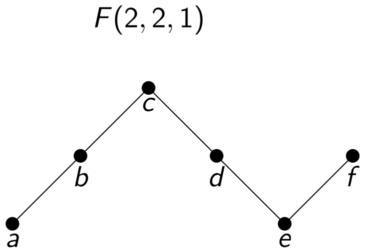

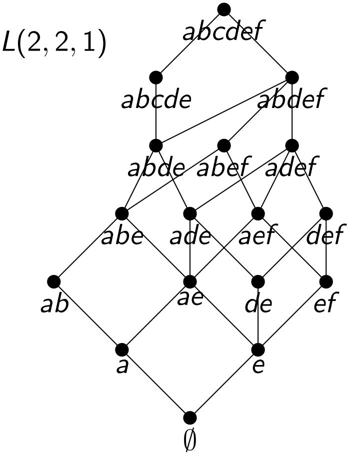

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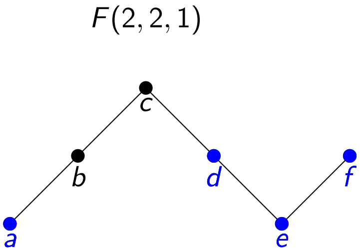

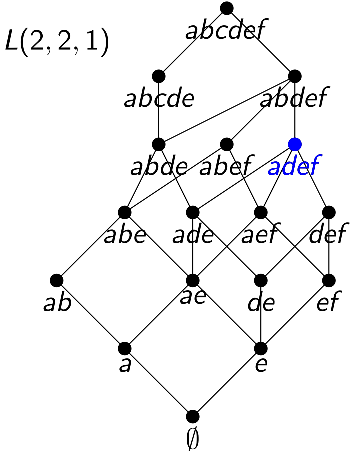

The lattices $L(\beta)$

The lattices $L(\beta)$ can be used to calculate the mutations in a cluster algebra derived from a surface with marked points on the boundary [Schiffler '08 '10, Schiffler–Thomas '09, Musiker–Schiffler–Williams '11, Yurikusa '19, Claussen '20, Propp '20].

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Since $L(\beta)$ is ranked, it has an associated rank sequence

$$r(\beta) : r_0, r_1, \ldots, r_n$$

where

$$\begin{aligned} r_k &= \text{number of elements at rank } k \text{ in } L(\beta) \\ &= \text{number of ideals of } F(\beta) \text{ of size } k. \end{aligned}$$

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The corresponding rank generating functions

$$r(q;\beta) = \sum_{k=0}^{n} r_k q^k$$

were used by Morier-Genoud and Ovsienko '20 to define q-analogues of rational and real numbers.

The rank generating function: example

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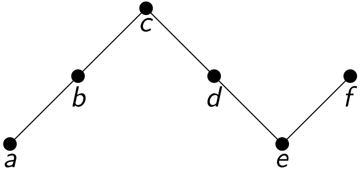

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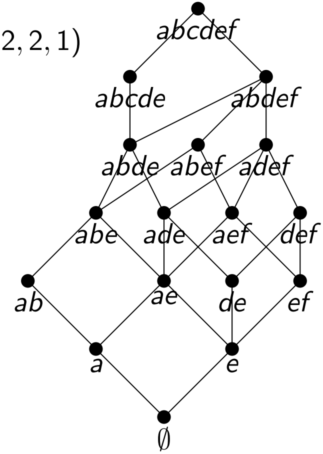

r_k
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4
4
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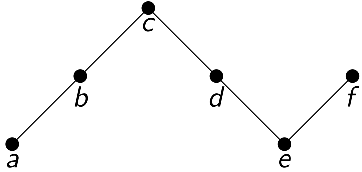

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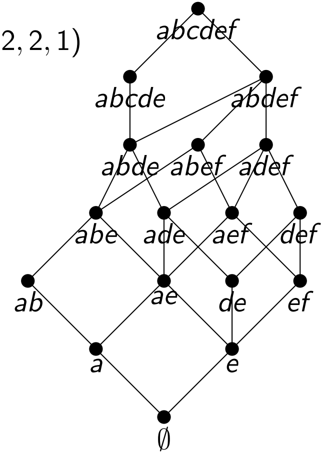

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The rank generating function: a larger example

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Definition

A sequence $r_0, r_1, \ldots, r_n$ is unimodal if there is an index m such that

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Conjecture (Morier-Genoud, Ovsienko '20)

For all β, the sequence $r(\beta)$ is unimodal.

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bottom interlacing $\quad \Longrightarrow \quad$ bottom heavy and unimodal

A refined conjecture

Conjecture (McConville, Sagan, Smyth '21)

Let $\beta = (\beta_1, \ldots, \beta_s)$.

- If $s = 1$ then $r(\beta) = (1, 1, \ldots, 1)$.
- If s is even, then $r(\beta)$ is bottom interlacing.
- Suppose $s \geq 3$ is odd and let $\beta' = (\beta_2, \ldots, \beta_{s-1})$.
  - If $\beta_1 > \beta_s$ then $r(\beta)$ is bottom interlacing.
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  - If $\beta_1 = \beta_s$ then $r(\beta)$ is symmetric, bottom interlacing, or top interlacing depending on whether $r(\beta')$ is symmetric, top interlacing, or bottom interlacing, respectively.

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The proof uses induction and algebraic manipulation, as well as a circular version of fences.

Our main result

In general, the sequence $r(\beta)$ is not symmetric, but we will show that it is exhibits partial symmetry.

Theorem

Suppose that $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ *where* s *is odd. Then, for all* $k \leq \min\{\beta_1, \beta_s\}$ *we have*

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Fix $\beta = (\beta_1, \beta_2, \dots, \beta_{2\ell+1})$ with an odd number of parts.

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To give a bijective proof of our main result, we will construct a bijection

$$\Phi : \mathcal{I}_k(\beta) \to \mathcal{U}_k(\beta)$$

for all $k \leq \min\{\beta_1, \beta_{2\ell+1}\}$.

A simpler case: gates

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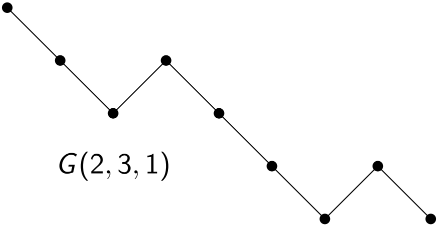
For a composition $\delta = (\delta_1, \delta_2, \ldots, \delta_\ell)$, define the gate

$$G(\delta) = F(\delta_1, 1, \delta_2, 1, \ldots, \delta_{\ell-1}, 1, \delta_\ell)^*,$$

where * indicates poset dual.



Restricted ideals of gates

Let $D_1, \ldots, D_\ell$ be the descending segments of $G(\delta)$ from left to right.

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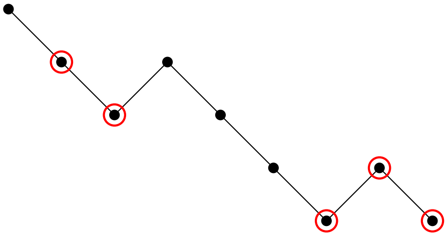
$I =$ 

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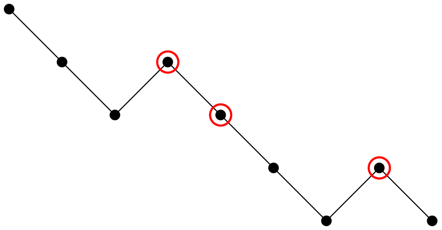

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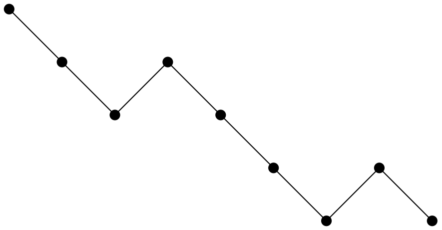
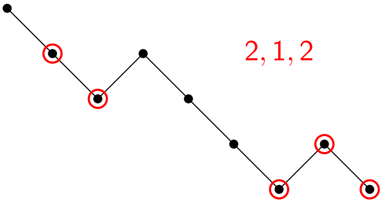


We will describe a cardinality-preserving bijection

$$\phi : \{\text{restricted ideals of } G(\delta)\} \to \{\text{restricted filters of } G(\delta)\}.$$

Encoding restricted ideals/filters of gates

A restricted ideal I of $G(\delta)$ can be encoded by a sequence $d_1, d_2, \ldots, d_\ell$, where $d_i = |I \cap D_i|$.



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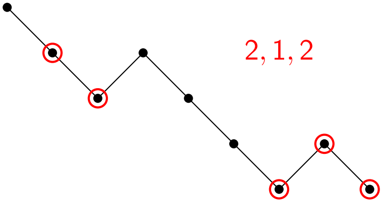

Such sequences can be characterized as those satisfying:

1. for $i \in [\ell]$, we have $0 \leq d_i \leq |D_i|$,
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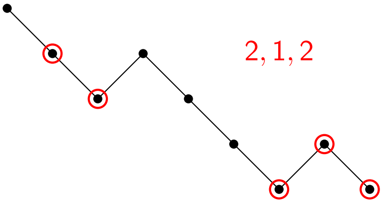

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Similarly, a restricted filter U of $G(\delta)$ can be encoded by a sequence $e_1, e_2, \ldots, e_\ell$, where $e_i = |U \cap D_i|$, characterized by similar conditions.

The bijection ϕ for restricted ideals/filters of gates

Given a sequence $d_1, d_2, \ldots, d_\ell$ encoding a restricted ideal I:

$$6, 1, 1, 1, 0, 4, 5, 1, 1, 0, 0, 3, 1, 2$$

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3. For each B' with $|B'| \geq 2$, decrease its rightmost entry by 1 and increase its leftmost entry by 1.

$$6, 1, 1, 1, 0, 4, 5, 1, 1, 0, 0, 3, 1, 2$$
$$6, 1, 1, 1, 0, 4, 5, 1, 1, 0, 0, 3, 1, 2$$
$$6, 1, 1, 1, 0, 4, 5, 1, 1, 0, 0, 3, 1, 2$$
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The bijection ϕ for restricted ideals/filters of gates

Given a sequence $d_1, d_2, \ldots, d_\ell$ encoding a restricted ideal I:

1. For each maximal block (consecutive subsequence) B of positive integers, factor it as $B = B'T$, where T is the maximal suffix consisting of 1s.
2. For each nonempty T, exchange T with the 0 to its right.
3. For each B' with $|B'| \geq 2$, decrease its rightmost entry by 1 and increase its leftmost entry by 1.

The resulting sequence encodes a restricted filter $\phi(I)$.

$$6, 1, 1, 1, 0, 4, 5, 1, 1, 0, 0, 3, 1, 2$$
$$6, 1, 1, 1, 0, 4, 5, 1, 1, 0, 0, 3, 1, 2$$
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The general case: encoding ideals of fences

Fix $\beta = (\beta_1, \beta_2, \ldots, \beta_{2\ell+1})$, and let $F = F(\beta)$.

Ascending segments: $A_1, A_2, \ldots, A_{\ell+1}$.
Descending segments: $D_1, D_2, \ldots, D_\ell$.

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Let $\tilde{A}_i$ be obtained from A_i by removing the elements shared with descending segments, so that each element appears in exactly one of the $\tilde{A}_i$ or D_i.

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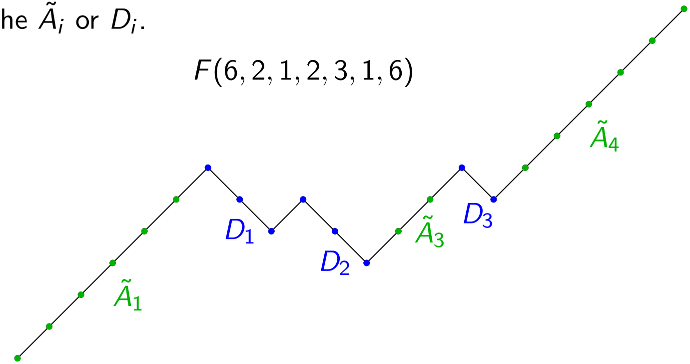
$F(6, 2, 1, 2, 3, 1, 6)$

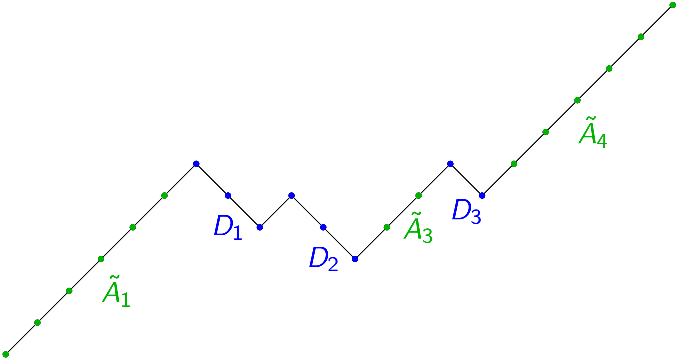


The general case: encoding ideals of fences

We encode ideals I of F as arrays of numbers

$$\begin{array}{ccccccccc} a_1 & & a_2 & & \cdots & & a_\ell & & a_{\ell+1} \\ & d_1 & & d_2 & & \cdots & & d_\ell & \end{array}$$

where $a_i = |I \cap \tilde{A}_i|$ and $d_i = |I \cap D_i|$ for all i.



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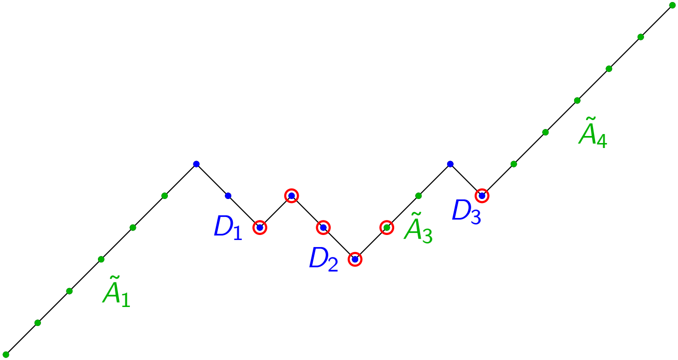
$I =$



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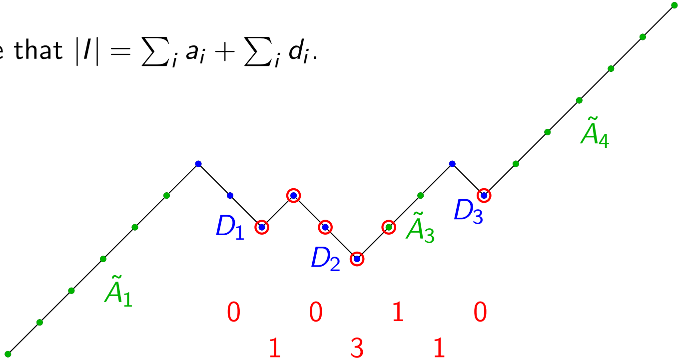
where $a_i = |I \cap \tilde{A}_i|$ and $d_i = |I \cap D_i|$ for all i.

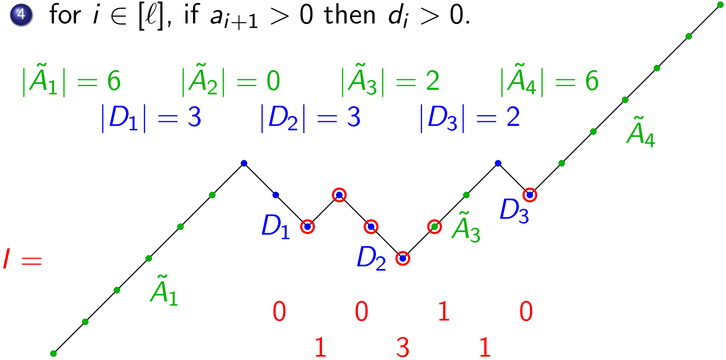
Note that $|I| = \sum_i a_i + \sum_i d_i$.

$I =$ 

The general case: encoding ideals of fences

Such an array encodes an ideal of F if and only if:

1. for $i \in [\ell + 1]$ we have $0 \le a_i \le |\tilde{A}_i|$,
2. for $i \in [\ell]$ we have $0 \le d_i \le |D_i|$,
3. for $i \in [\ell]$, if $d_i = |D_i|$ then $a_i = |\tilde{A}_i|$, and if $i > 1$ then $d_{i-1} > 0$ as well,
4. for $i \in [\ell]$, if $a_{i+1} > 0$ then $d_i > 0$.



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Similarly, we encode filters U of F as arrays of numbers

$$\begin{array}{ccccccccc} b_1 & & b_2 & & \cdots & & b_\ell & & b_{\ell+1} \\ & e_1 & & e_2 & & \cdots & & e_\ell & \end{array}$$

where $b_i = |U \cap \tilde{A}_i|$ and $e_i = |U \cap D_i|$ for all i.

Such arrays can be characterized by similar conditions.

The bijection Φ for ideals/filters of fences

Next we define $\Phi : \mathcal{I}_k(F) \to \mathcal{U}_k(F)$, where $k \leq \min\{\beta_1, \beta_{2\ell+1}\}$.

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$$\begin{array}{ccccccc} 6 & & 0 & & 2 & & 6 \\ \vee\!| & & \vee\!| & & \vee\!| & & \vee\!| \\ 0 & & 0 & & 1 & & 0 \\ & 1 & & 3 & & 1 & \end{array}$$

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$$\begin{array}{ccccccc} 6 & & 0 & & 2 & & 6 \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ 0 & & 0 & & 1 & & 1 \\ & 1 & & 3 & \mid & 0 & \end{array}$$

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$$\begin{array}{ccccccc} 6 & & 0 & & 2 & & 6 \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ 0 & & 0 & & 1 & & 1 \\ & 1 & & 3 & | & 0 & \end{array}$$

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$$\begin{matrix} a_1 & & a_2 & & \cdots & & a_\ell & & a_{\ell+1} \\ & d_1 & & d_2 & & \cdots & & d_\ell & \end{matrix}$$

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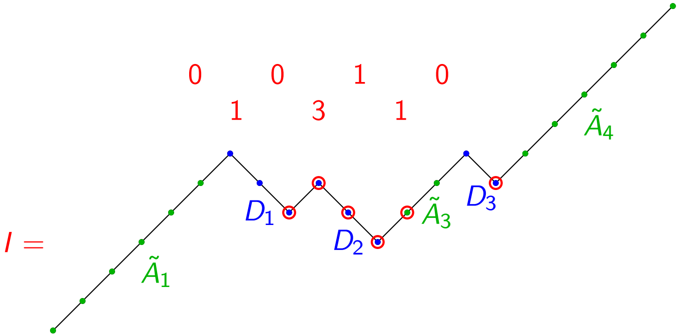
3. For every i s.t. $b_i > 0$ and $e_i = 0$, let $b_i := b_i - 1$ and $e_i := 1$.

The resulting array encodes the filter $\Phi(I)$.

$$\begin{array}{ccccccc} 6 & & 0 & & 2 & & 6 \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ 0 & & 0 & & 0 & & 1 \\ & 2 & & 2 & & 1 & \end{array}$$

Example of Φ



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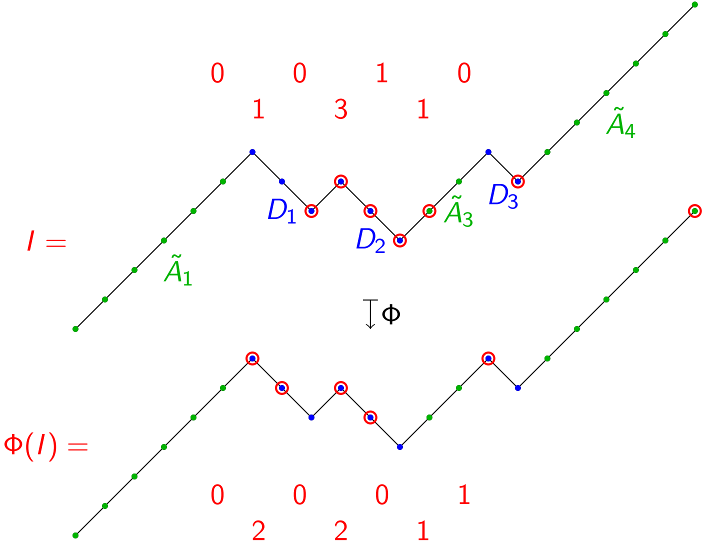
0 0 1 0
1 3 1
$\tilde{A}_4$
D_1
D_2
$\tilde{A}_3$
D_3
$I =$
$\tilde{A}_1$
$\mapsto \Phi$
$\Phi(I) =$
0 0 0 1
2 2 1

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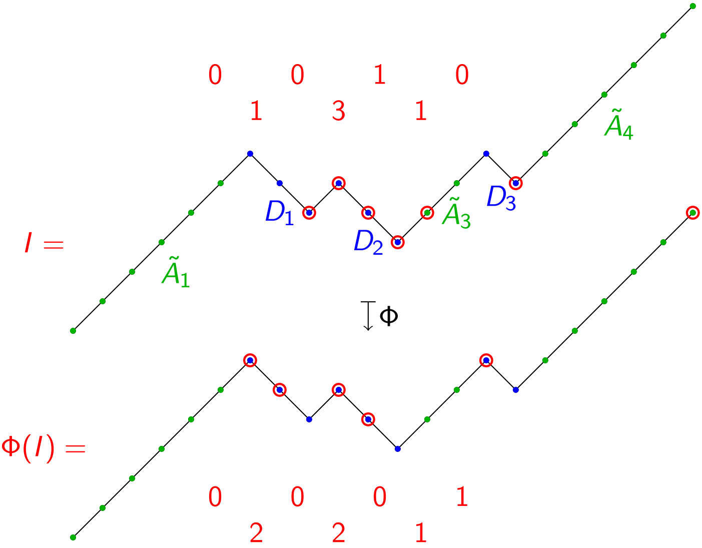

The inverse Φ^{-1} is essentially Φ conjugated with 180° rotation.

Circular fences

Oğuz and Ravichandran's proof of the (refined) unimodality of the sequences $r(\beta)$ relies on so-called circular fences.

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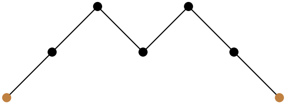
Let $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$, where s is even.

Definition

The circular fence $\overline{F}(\beta)$ is obtained by identifying the leftmost and the rightmost elements of $F(\beta)$.

$F(2,1,1,2)$



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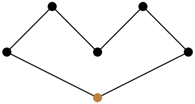
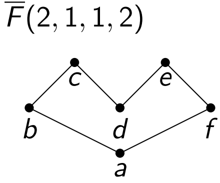
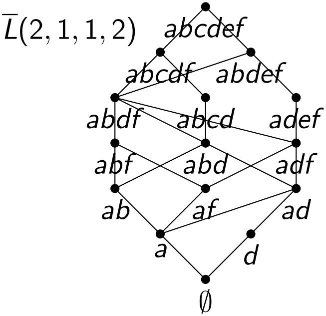


Rank symmetry for circular fences

Let $\overline{L}(\beta)$ be the distributive lattice of lower order ideals of $\overline{F}(\beta)$, ordered by containment, and let $\overline{r}(\beta)$ be its rank sequence.





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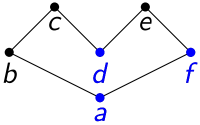

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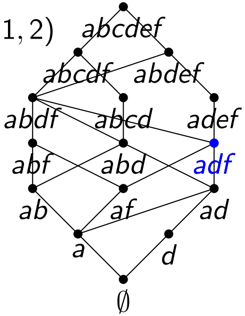

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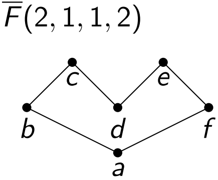

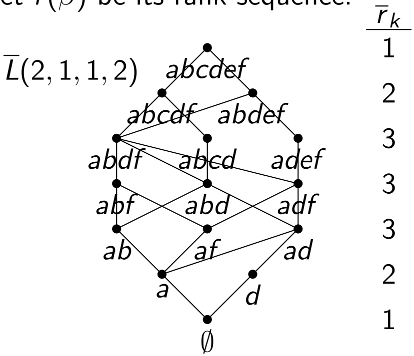

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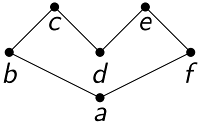

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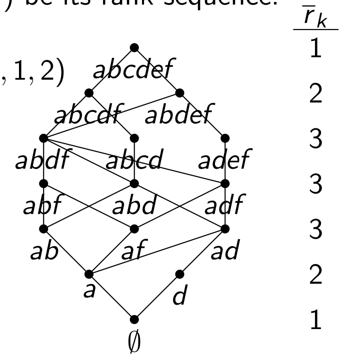

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For every circular fence, the sequence $\overline{r}(\beta)$ is symmetric.

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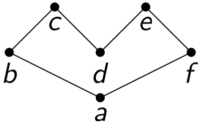

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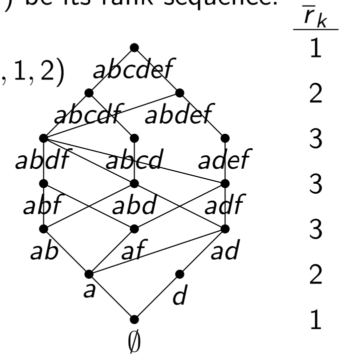

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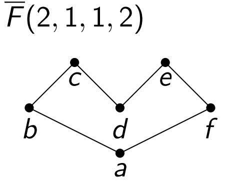

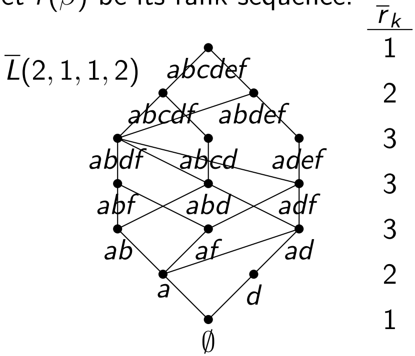

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We will give a bijective proof by modifying our bijection for fences.

Ideals and filters of circular fences: easier case

We want a cardinality-preserving bijection

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First consider the case where ascending segments have length 1. Let $\beta = (1, \delta_1, 1, \delta_2, \ldots, 1, \delta_\ell)$, and let $D_1, \ldots, D_\ell$ be the descending segments of $\overline{F}(\beta)$.

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An ideal I of $\overline{F}(\beta)$ can be encoded by a sequence $d_1, d_2, \ldots, d_\ell$, where $d_i = |I \cap D_i|$, satisfying:

1. for $i \in [\ell]$ we have $0 \le d_i \le |D_i|$,
2. for $i \in [\ell]$, if $d_i = |D_i|$ then $d_{i-1} > 0$, with subscripts modulo ℓ.

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Similarly, a filter U of $\overline{F}(\beta)$ can be encoded by a sequence $e_1, e_2, \ldots, e_\ell$, where $e_i = |U \cap D_i|$, satisfying analogous conditions.

The bijection $\overline{\phi}$ for $\overline{F}(1, \delta_1, 1, \delta_2, \dots, 1, \delta_\ell)$

If $d : d_1, d_2, \dots, d_\ell$ encodes an ideal I, denote by $\langle d \rangle$ the circular sequence with subscripts taken modulo ℓ, so that d_ℓ and d_1 are considered adjacent.

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If all the entries of $\langle d \rangle$ are positive, do nothing. Otherwise:

1. For each maximal block B of positive integers in $\langle d \rangle$,

$$\langle 7, 1, 1, 0, 5, 1, 0, 0, 3 \rangle$$
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1. For each maximal block B of positive integers in $\langle d \rangle$, factor it as $B = B'T$, where T is the maximal suffix consisting of 1s.

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3. For each B' with $|B'| \geq 2$, decrease its last entry by 1 and increase its first entry by 1.

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$$\langle 6, 0, 1, 1, 5, 0, 1, 0, 4 \rangle$$

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The resulting sequence encodes the filter $\overline{\phi}(I)$.

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Encoding ideals/filters of arbitrary circular fences $\overline{F}(\beta)$

Consider now the general case, where $\beta = (\beta_1, \beta_2, \ldots, \beta_{2\ell})$.

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with subscripts modulo ℓ.

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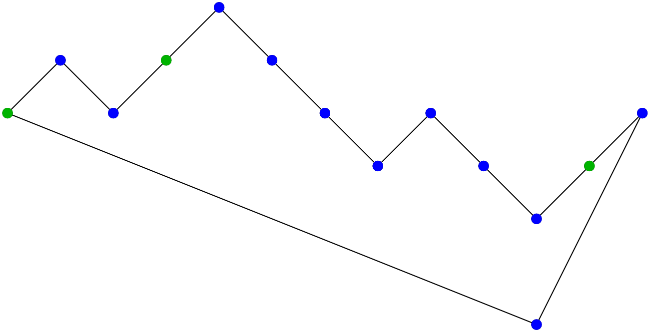
A filter of $\overline{F}(\beta)$ can be encoded similarly.

Encoding ideals/filters of arbitrary circular fences $\overline{F}(\beta)$

$\overline{F}(2, 1, 2, 3, 1, 2, 2, 1)$



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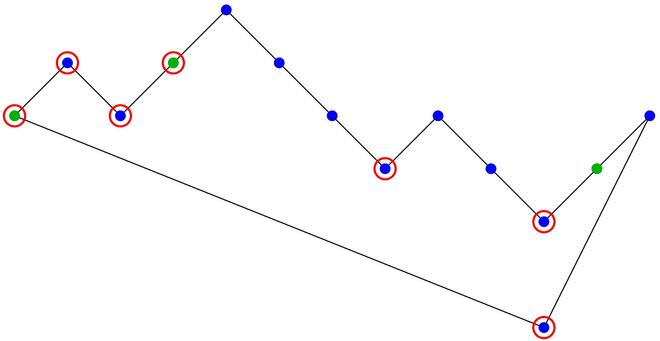

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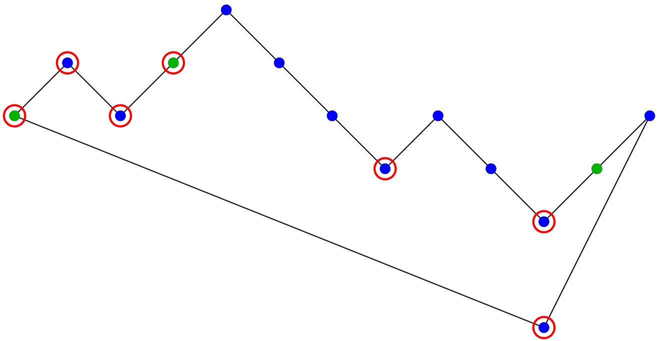

1		1		0		1		
$\vee\!\mid$		$\vee\!\mid$		$\vee\!\mid$		$\vee\!\mid$		
1		1		0		0		
	2		1		1		1	

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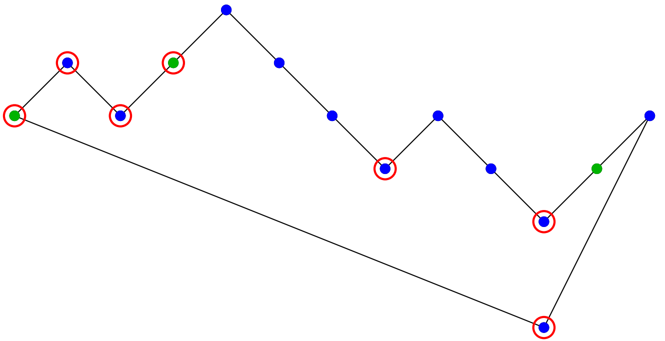

$$\begin{matrix} 1 & & 1 & & 0 & & 1 & & \\ \geq & & \geq & & \geq & & \geq & & \\ 1 & & 1 & & 0 & & 0 & & 1 \\ & 2 & & 1 & & 1 & & 1 & \end{matrix}$$

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$$\begin{array}{ccccccccc} \color{green}{1} & & \color{green}{1} & & \color{green}{0} & & \color{green}{1} & & \\ \color{green}{\geq} & & \color{green}{\geq} & & \color{green}{\geq} & & \color{green}{\geq} & & \\ 1 & & 1 & & 0 & & 0 & & 1 \\ & 2 & & 1 & & 1 & & 1 & \end{array} \qquad \begin{array}{ccccccccc} \color{green}{1} & & \color{green}{1} & & \color{green}{0} & & \color{green}{1} & & \\ \color{green}{\geq} & & \color{green}{\geq} & & \color{green}{\geq} & & \color{green}{\geq} & & \\ 1 & & 0 & & 0 & & 0 & & 1 \\ & 2 & & 1 & & 1 & & 1 & \end{array}$$

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3. For every $i \in [\ell]$ s.t. $b_i > 0$ & $e_i = 0$, let $b_i := b_i - 1$ & $e_i := 1$.

$$\begin{array}{ccccccccc} 1 & & 1 & & 0 & & 1 & & \\ \geq & & \geq & & \geq & & \geq & & \\ 1 & & 1 & & 0 & & 1 & & 1 \\ & 1 & & 0 & & 1 & & 2 & \end{array} \qquad \begin{array}{ccccccccc} 1 & & 1 & & 0 & & 1 & & \\ \geq & & \geq & & \geq & & \geq & & \\ 1 & & 0 & & 0 & & 1 & & 1 \\ & 1 & & 0 & & 1 & & 2 & \end{array}$$

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3. For every $i \in [\ell]$ s.t. $b_i > 0$ & $e_i = 0$, let $b_i := b_i - 1$ & $e_i := 1$.

$$\begin{array}{ccccccccc} 1 & & 1 & & 0 & & 1 & & \\ \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \\ 1 & & 0 & & 0 & & 1 & & 1 \\ & 1 & & 1 & & 1 & & 2 & \end{array} \qquad \begin{array}{ccccccccc} 1 & & 1 & & 0 & & 1 & & \\ \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \\ 1 & & 0 & & 0 & & 1 & & 1 \\ & 1 & & 0 & & 1 & & 2 & \end{array}$$

The bijection $\overline{\Phi}$ for arbitrary circular fences $\overline{F}(\beta)$

Suppose that an ideal I of $\overline{F}(\beta)$ is encoded by an array

$$\begin{matrix} a_1 & & a_2 & & \cdots & & a_\ell & & a_1 \\ & d_1 & & d_2 & & \cdots & & d_\ell & \end{matrix}$$

We perform the following operations, with subscripts modulo ℓ:

1. For every $i \in [\ell]$ s.t. $d_i = 1$ and $a_{i+1} < |\tilde{A}_{i+1}|$, let $d_i := 0$ and $a_{i+1} := a_{i+1} + 1$.
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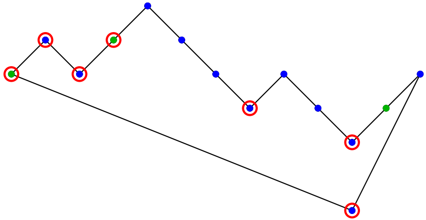
The resulting array encodes a filter $\overline{\Phi}(I)$.

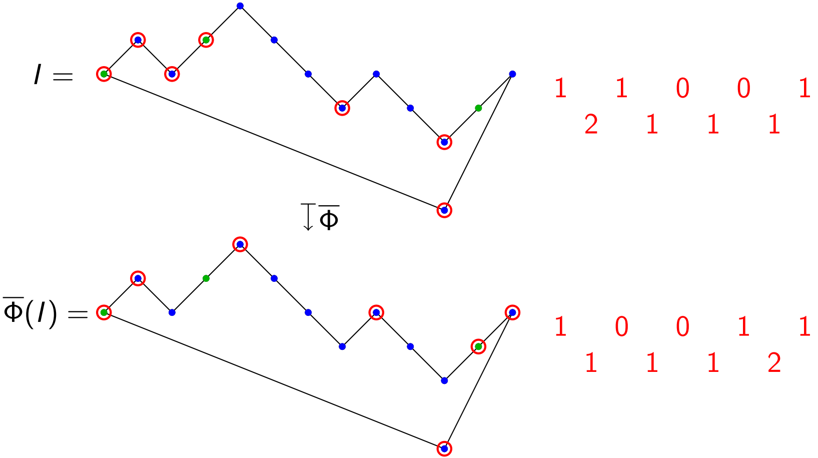
$$\begin{matrix} 1 & & 1 & & 0 & & 1 & & \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\ 1 & & 0 & & 0 & & 1 & & 1 \\ & 1 & & 1 & & 1 & & 2 & \end{matrix} \qquad \begin{matrix} 1 & & 1 & & 0 & & 1 & & \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\ 1 & & 0 & & 0 & & 1 & & 1 \\ & 1 & & 0 & & 1 & & 2 & \end{matrix}$$

Example of $\overline{\Phi}$

$I =$ 

$$\begin{matrix} 1 & & 1 & & 0 & & 0 & & 1 \\ & 2 & & 1 & & 1 & & 1 & \end{matrix}$$

Example of $\overline{\Phi}$



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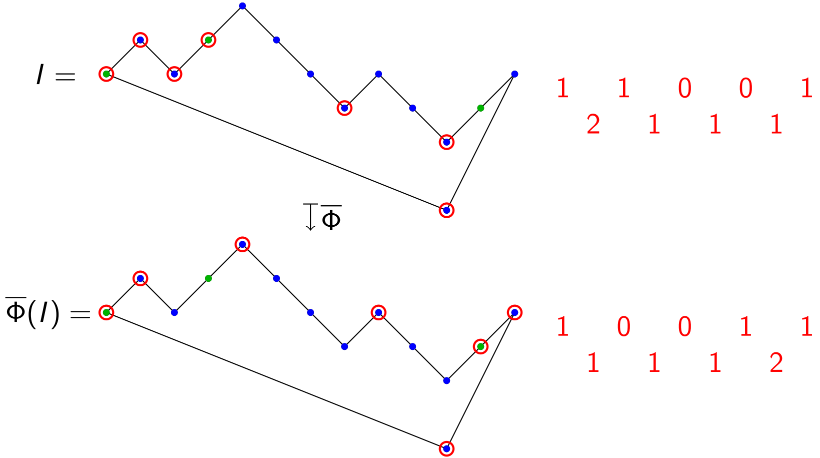
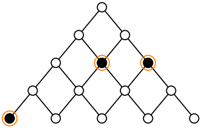


The inverse map $\overline{\Phi}^{-1}$ can be described by applying $\overline{\Phi}$ to the horizontal reflection of the arrays.

Rowmotion on antichains of a poset

antichains



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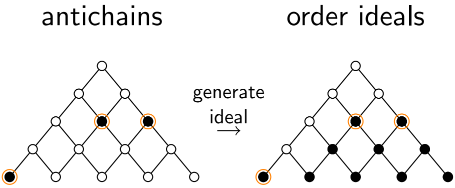

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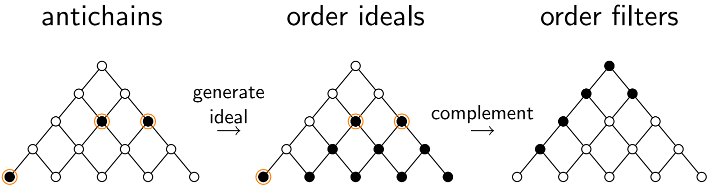
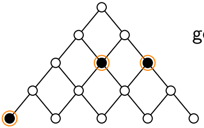
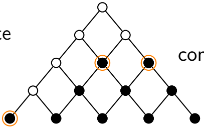


Rowmotion on antichains of a poset

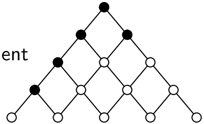
antichains | order ideals | order filters



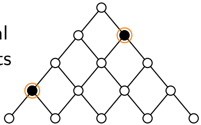
generate ideal $\longrightarrow$



complement $\longrightarrow$



minimal elements $\longrightarrow$



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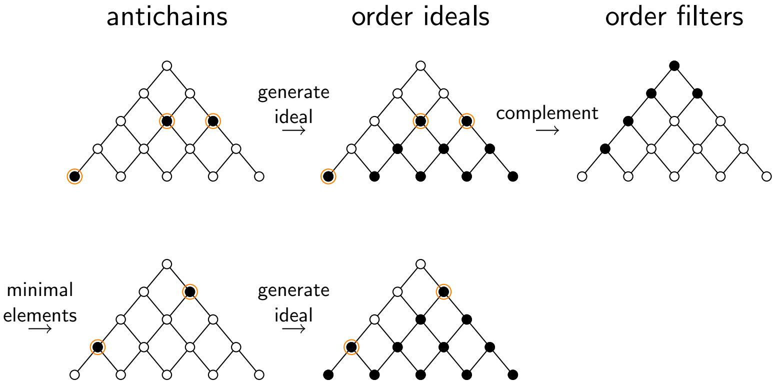

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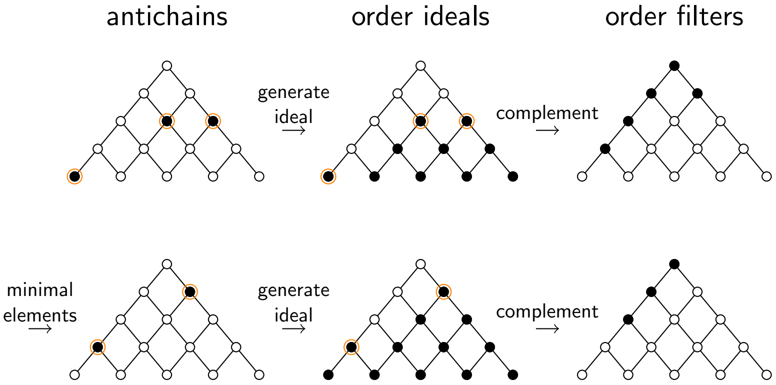

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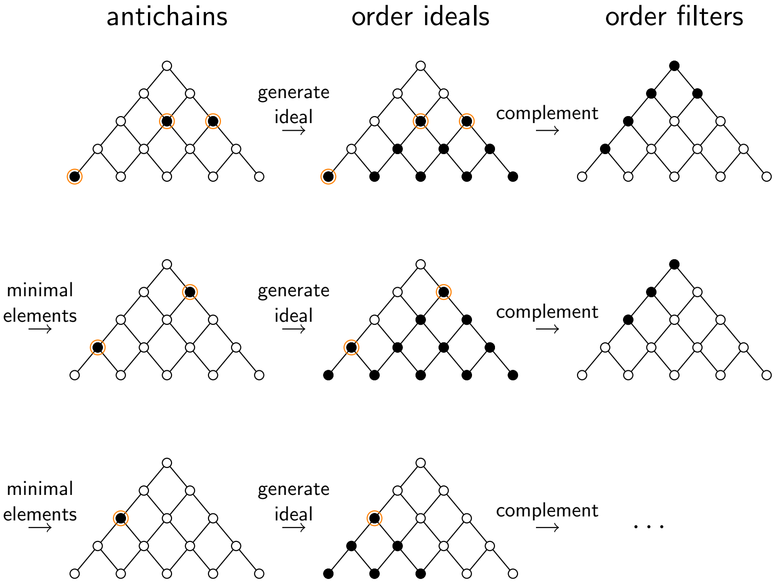

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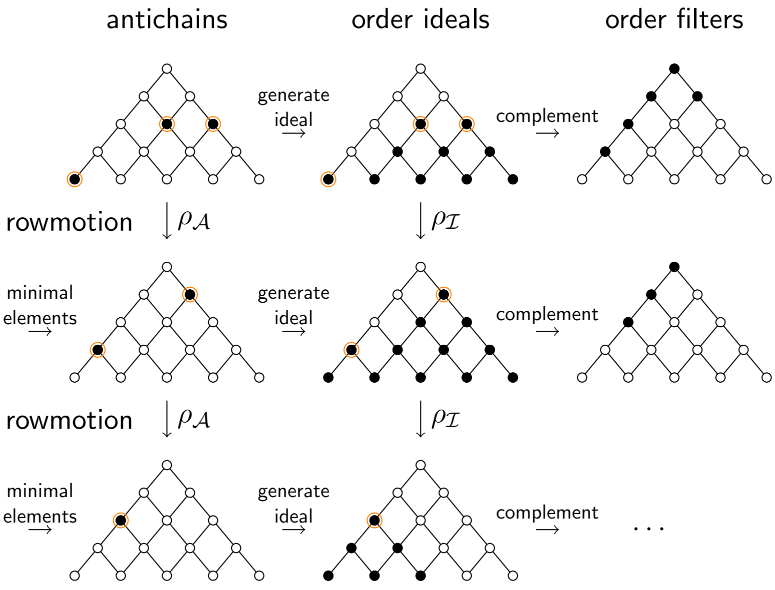

Rowmotion, homomesy and homometry

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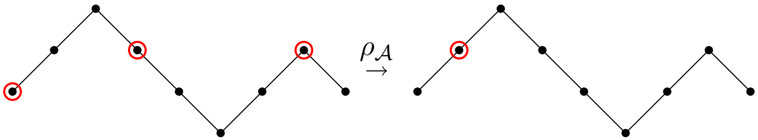
Rowmotion on fences



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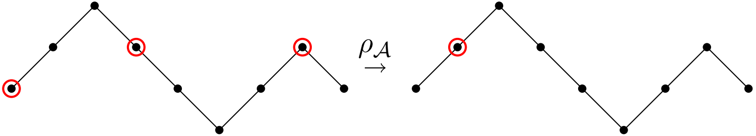

Theorem (E.–Plante–Roby–Sagan '21)

For fences with two segments $F(a-1, b-1)$:

- *rowmotion has $\gcd(a,b)$ orbits, of which all have size $\operatorname{lcm}(a,b)$ except for one that has size $\operatorname{lcm}(a,b)+1$.*

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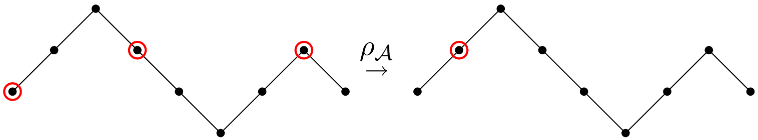

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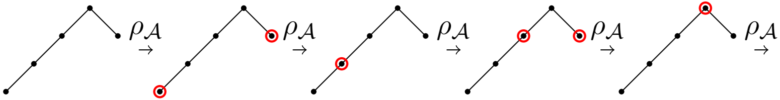

Rowmotion on fences

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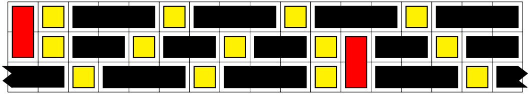

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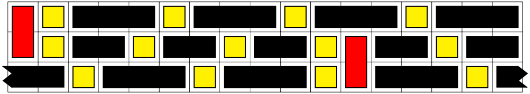

Conjecture (E.–Plante–Roby–Sagan '21)

For fences of the form $F(a-1, a, a, \ldots, a, a-1)$:

- the statistic $\chi_{\mathcal{A}}$ is homometric under the action of $\rho_{\mathcal{A}}$,
- if the number of segments is odd, the statistic $\chi_{\mathcal{I}}$ is homomesic under the action of $\rho_{\mathcal{I}}$.

Open questions

For fences $F(\beta)$, Oğuz and Ravichandran proved recursively that the sequences $r(\beta)$ are unimodal and, more strongly, bottom or top interlacing depending on the case.

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Can one modify the bijection Φ to give an injective proof?

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In the case of circular fences $\overline{F}(\beta)$, unimodality of $\overline{r}(\beta)$ does not always hold, but it often does.

Conjecture (Oğuz–Ravichandran '21)

Assuming β has an even number of parts, $\overline{r}(\beta)$ is unimodal except when $\beta = (1, k, 1, k)$ or $\beta = (k, 1, k, 1)$ for some $k \geq 1$.

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Question 2

Can this be proved by modifying the bijection $\overline{\Phi}$?

Open problems

Recall that $a_0, a_1, \ldots, a_n$ is log-concave if

$$a_i^2 \geq a_{i-1}a_{i+1}$$

for all $0 < i < n$. For positive sequences, this condition implies unimodality.

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For which β are $r(\beta)$ or $\overline{r}(\beta)$ log-concave?

The sequences $r(\beta)$ are not always log-concave, e.g. $r(1,1): 1, 2, 1, 1$.

The sequences $\overline{r}(\beta)$ can be unimodal but not log-concave, e.g. $\overline{r}(1,1,1,1,1,1): 1, 3, 3, 4, 3, 1$.

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For any poset P, denote by $L(P)$ its lattice of order ideals.

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THE END

- Elizalde and Sagan, Partial rank symmetry of distributive lattices for fences, arXiv:2201.03044.
- Elizalde, Plante, Roby and Sagan, Rowmotion on fences, arXiv:2108.12443.