Forbidden patterns in telling random from deterministic time series

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Joint work with José M. Amigó and Matt Kennel

Deterministic or random?

Two sequences of numbers in $[0, 1]$:

$$.6416,\ .9198,\ .2951,\ .8320,\ .5590,\ .9861,\ .0550,\ .2078,\ .6584,\ .8996,\ .3612,\ .9230,\ .2844,\ .8141,\ .6054, \ldots$$

$$.9129,\ .5257,\ .4475,\ .9815,\ .4134,\ .9930,\ .1576,\ .8825,\ .3391,\ .0659,\ .1195,\ .5742,\ .1507,\ .5534,\ .0828, \ldots$$

Are they random? Are they deterministic?

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Are they random? Are they deterministic?

Let $f(x) = 4x(1 - x)$. Then

$f(.6146) = .9198$,

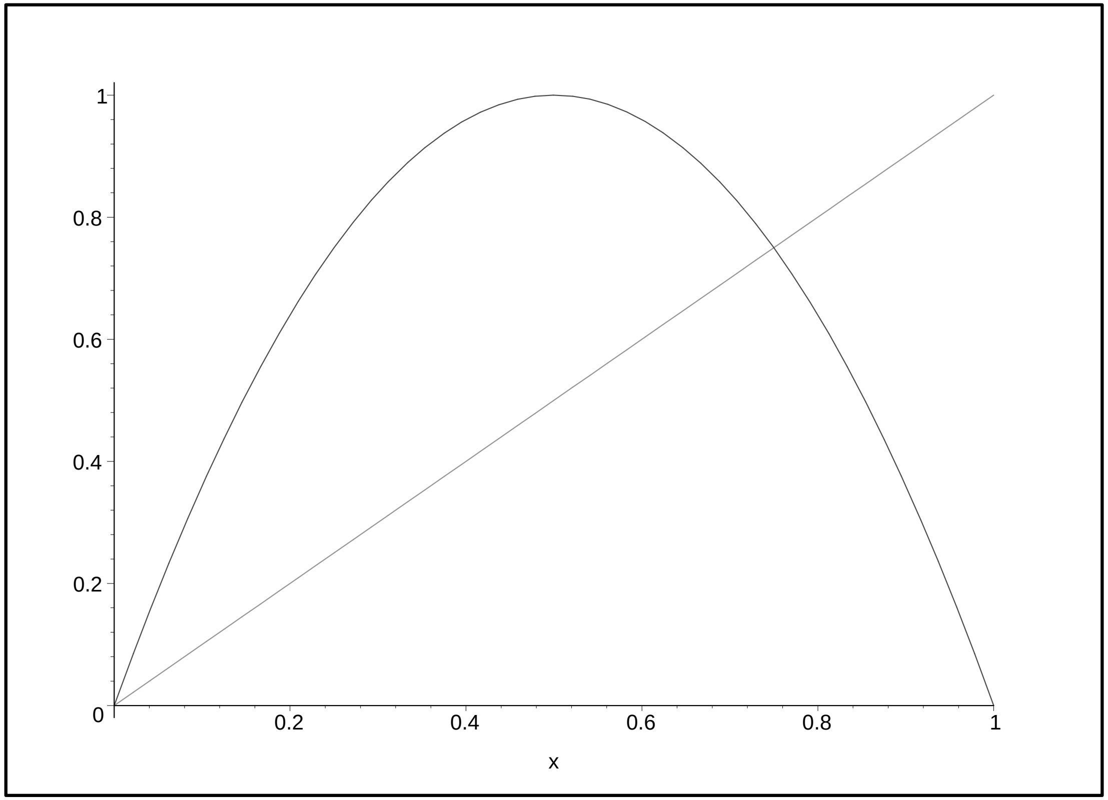
$f(.9198) = .2951$,

$f(.2951) = .8320$,

$\ldots$

Order patterns of a map

$$f : [0,1] \to [0,1], \quad f(x) = 4x(1-x)$$



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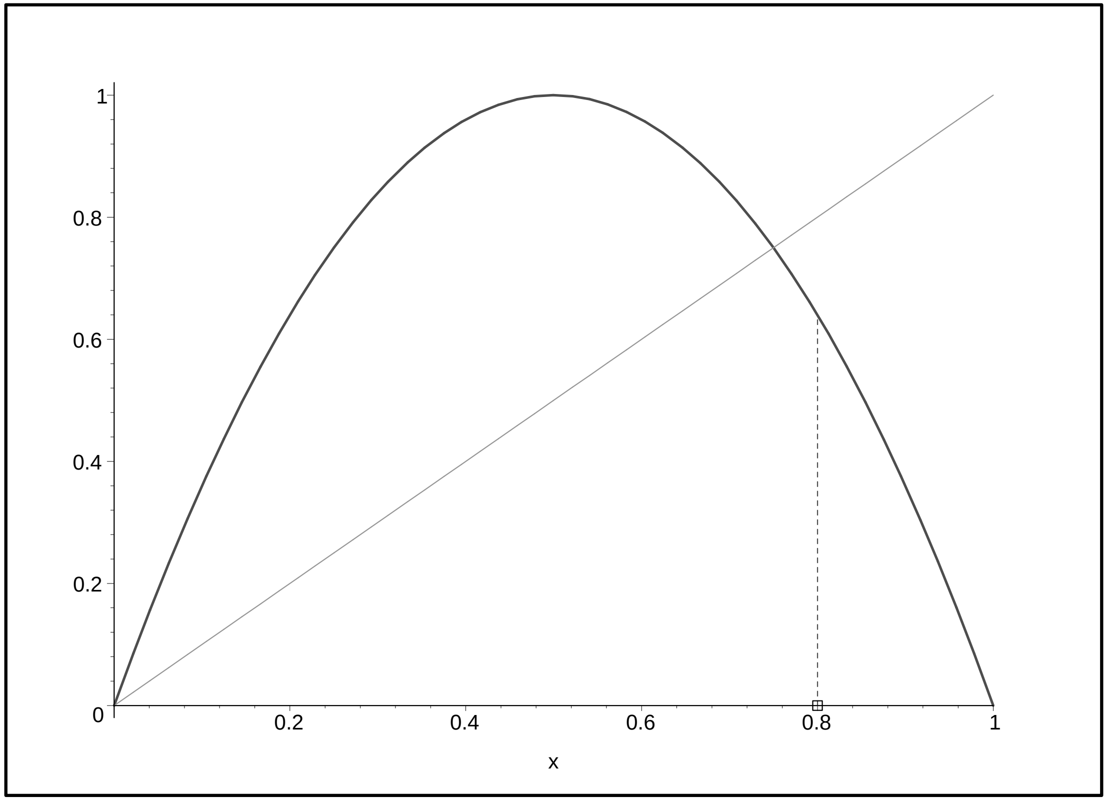

Given $x \in [0,1]$, consider the sequence

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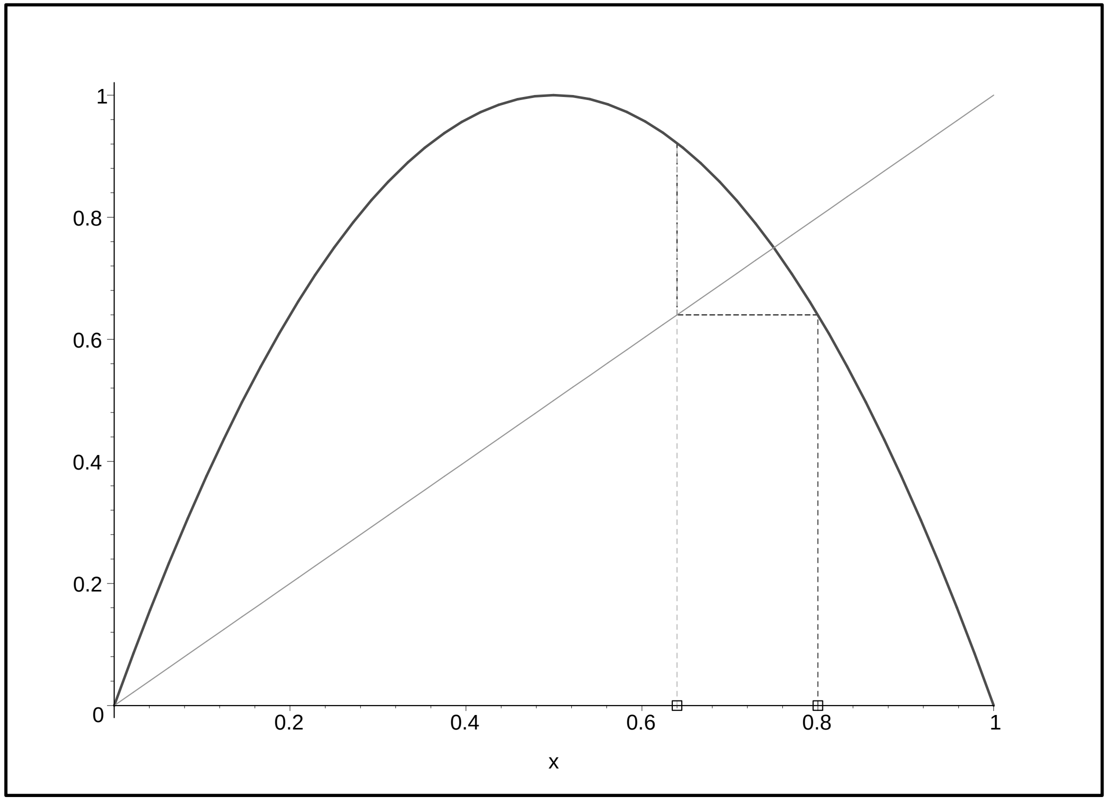

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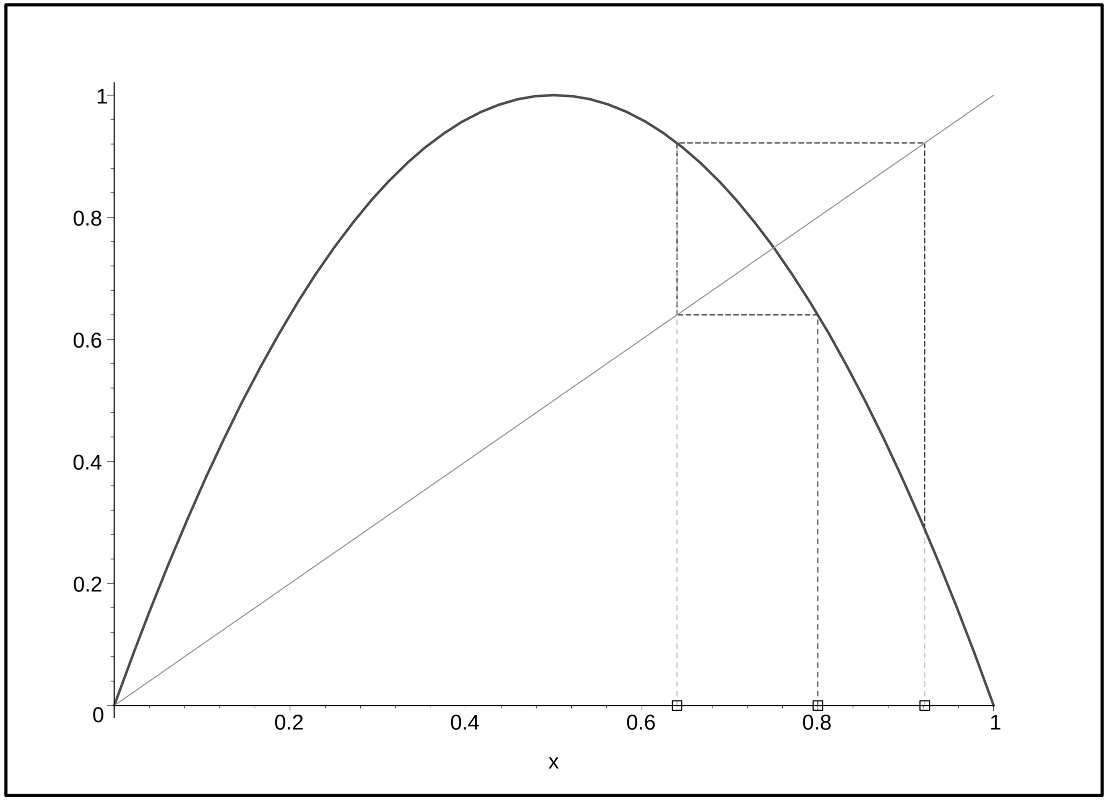

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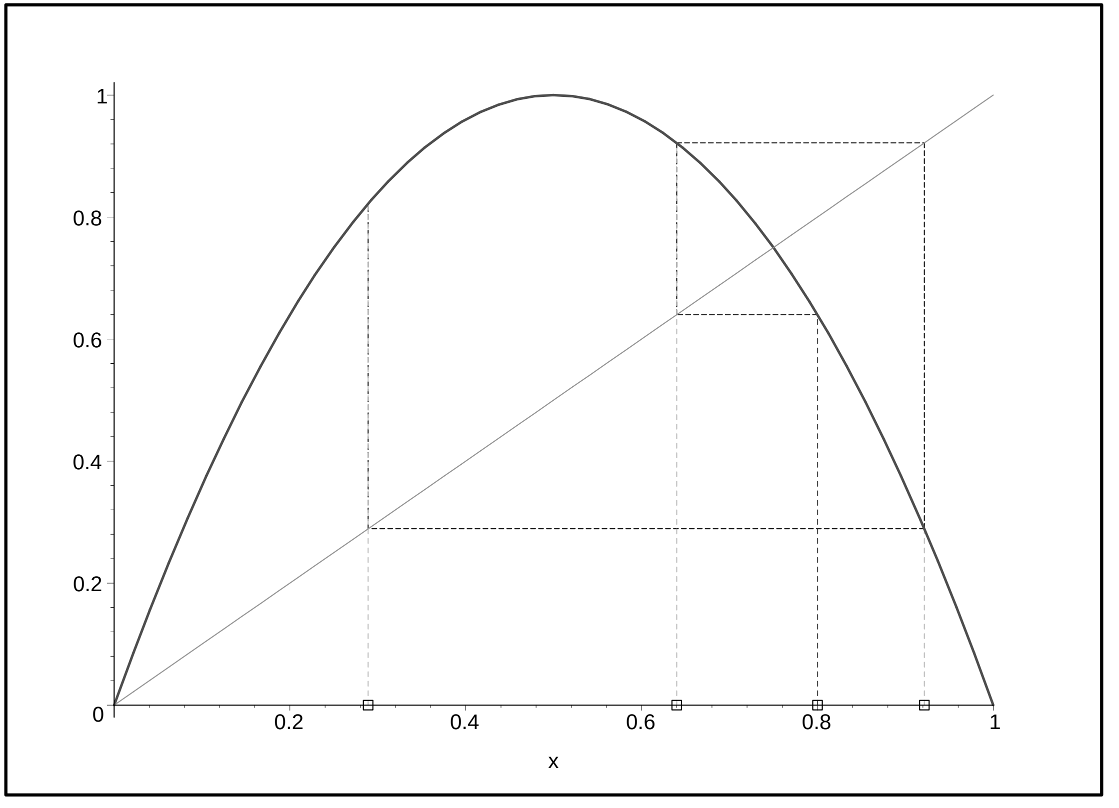

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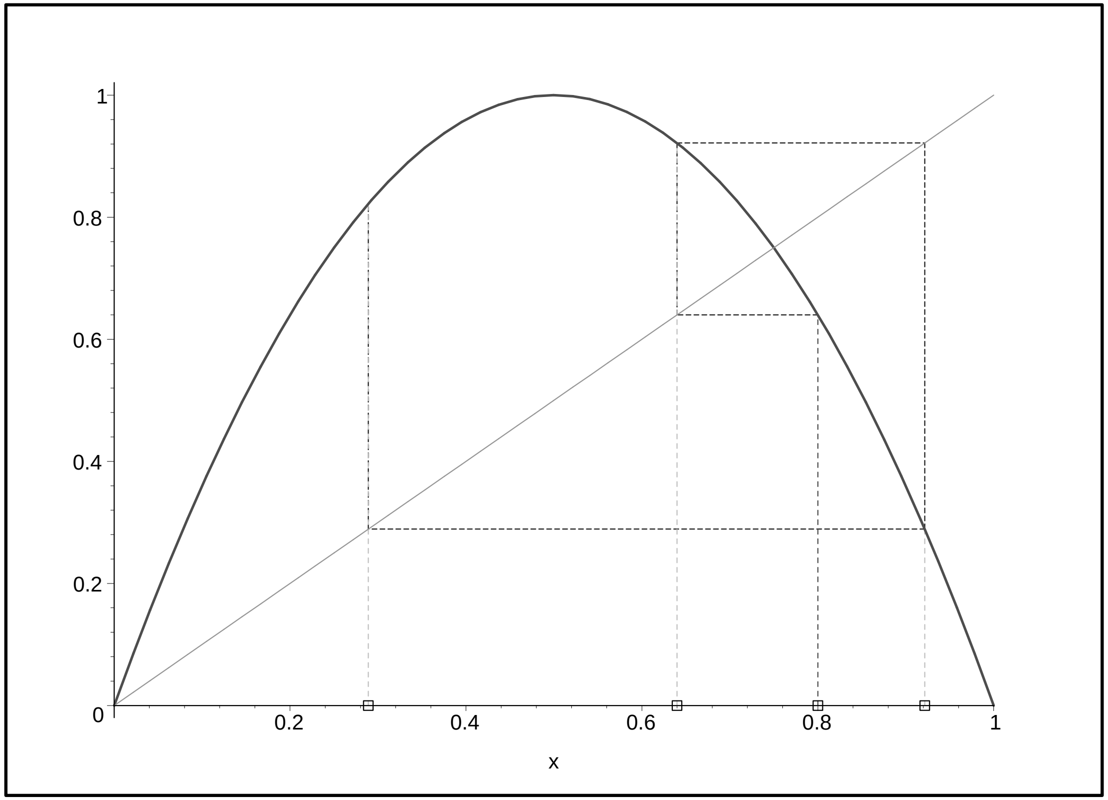

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We say that x *defines the order pattern* $[3, 2, 4, 1]$.

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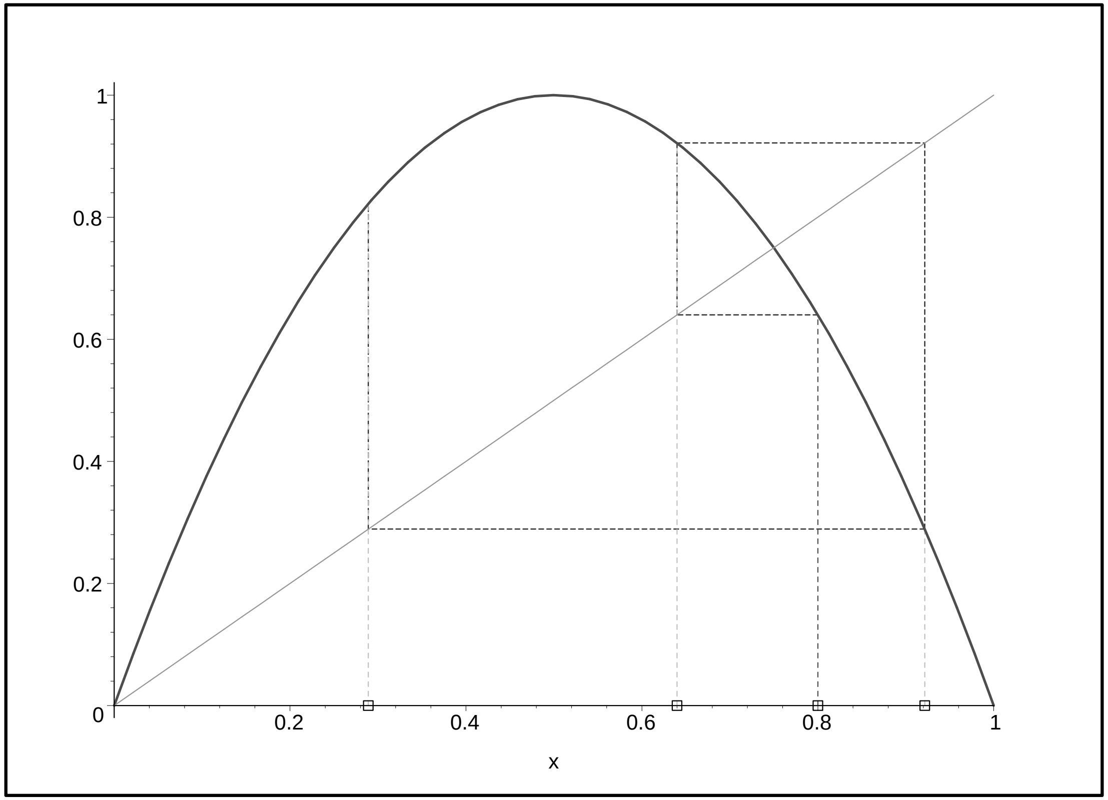

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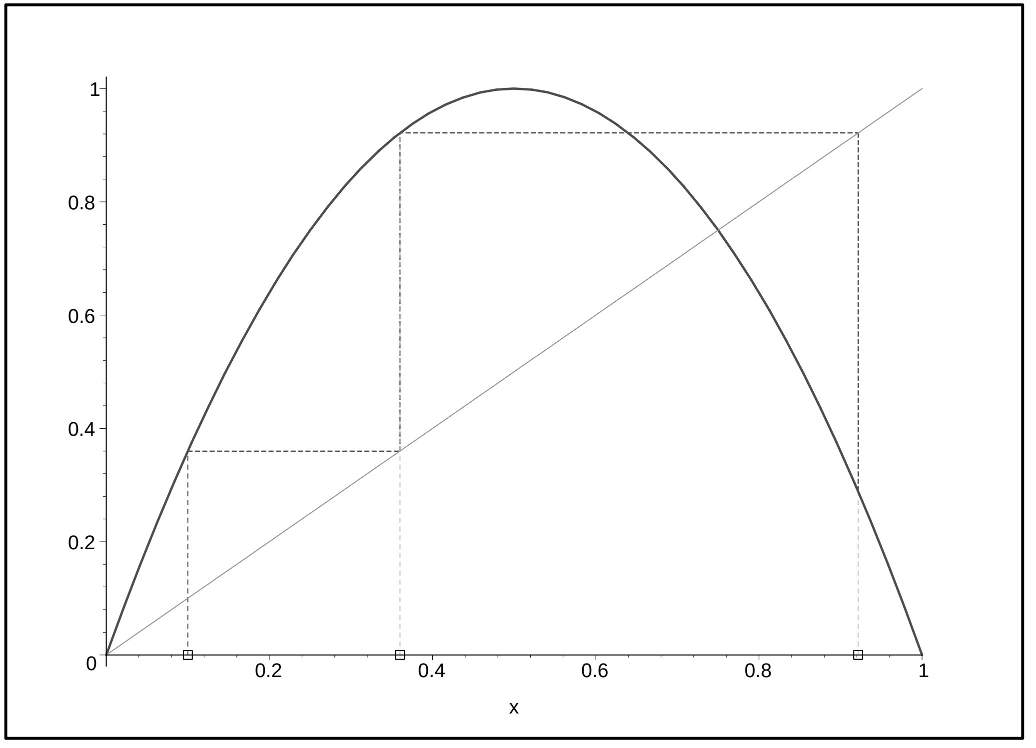
For $x = 0.8$ and $k = 4$, we get $[0.8,\ 0.64,\ 0.9216,\ 0.2890]$
We say that x *defines the order pattern* 3241.

What patterns can appear?

Let $k = 3$.



$x = 0.1$ defines the order pattern $\color{blue}{123}$

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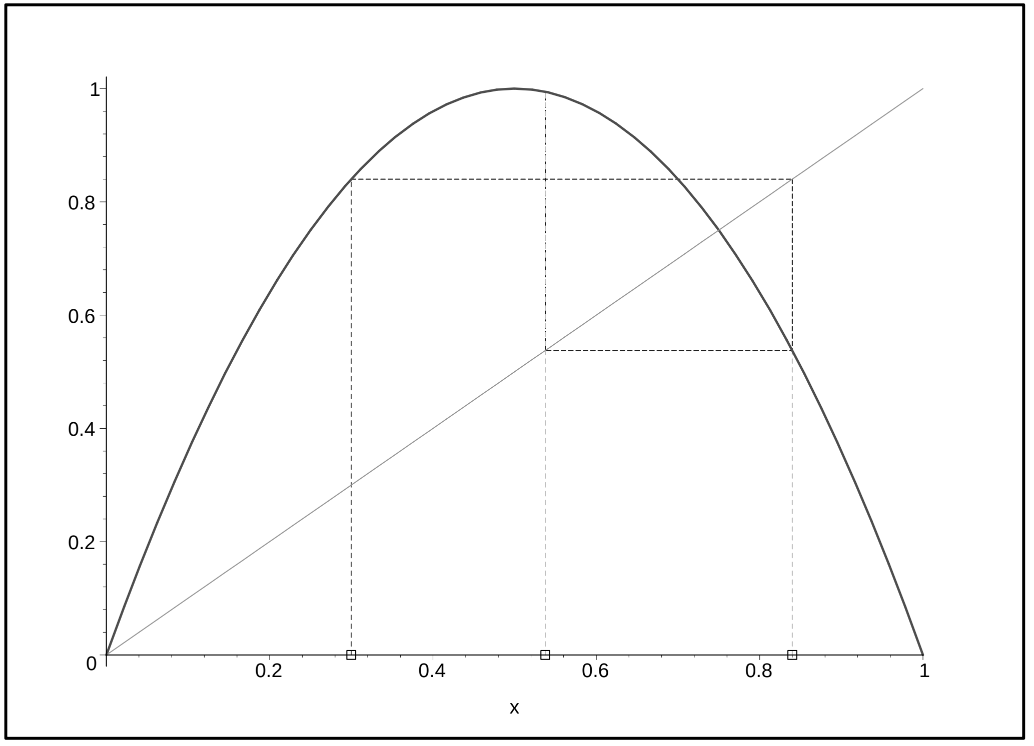
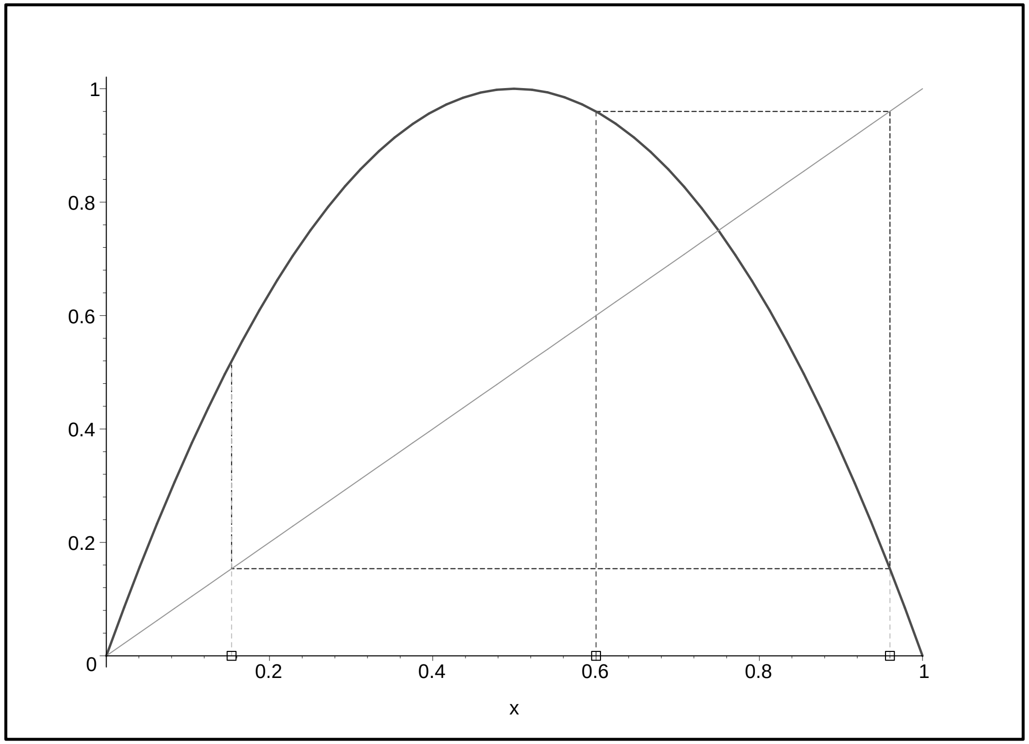


$x = 0.1$ defines the order pattern 123

$x = 0.3$ defines the order pattern 132

Let $k = 3$.



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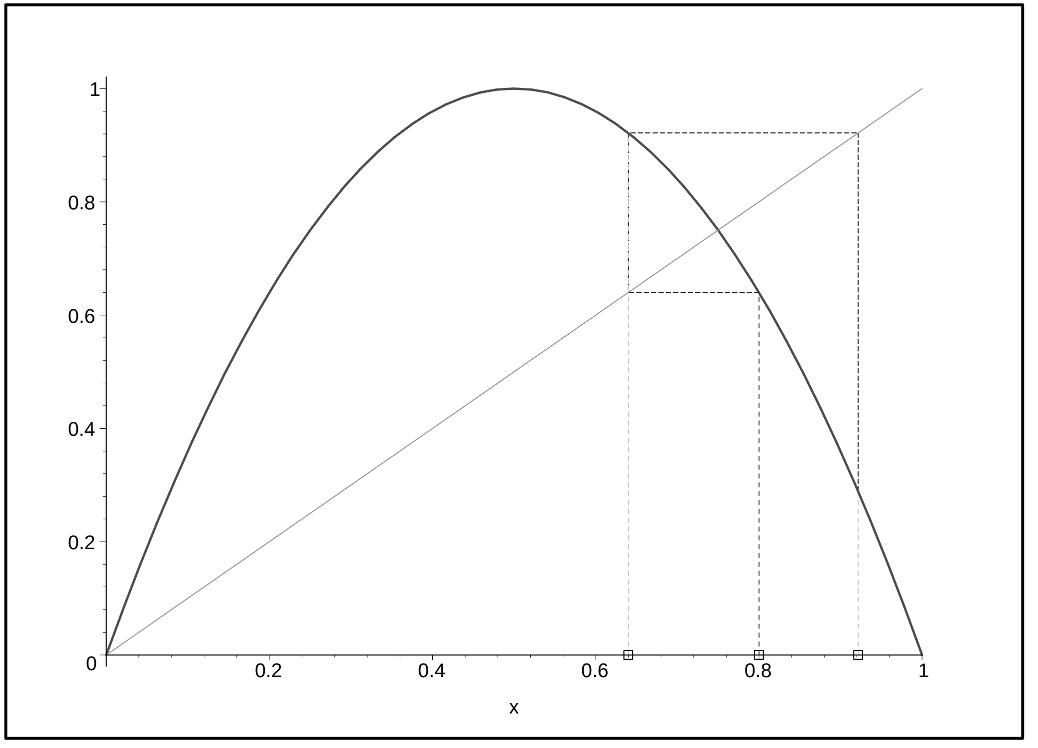

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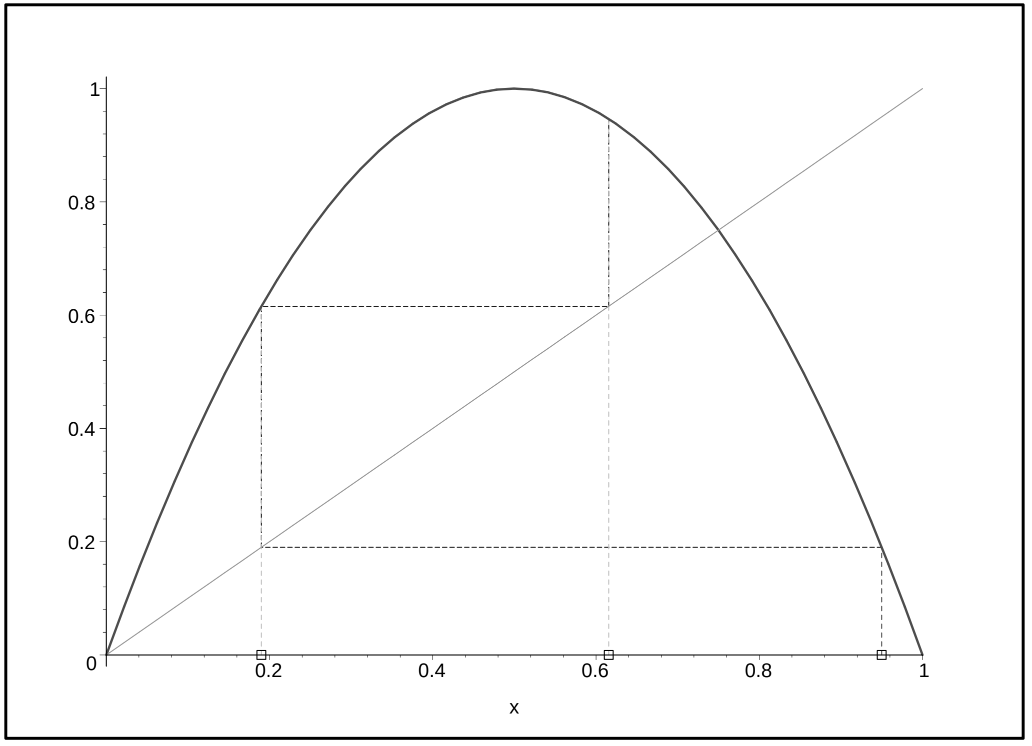

$x = 0.1$	defines the order pattern	123
$x = 0.3$	defines the order pattern	132
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$x = 0.95$	defines the order pattern	312

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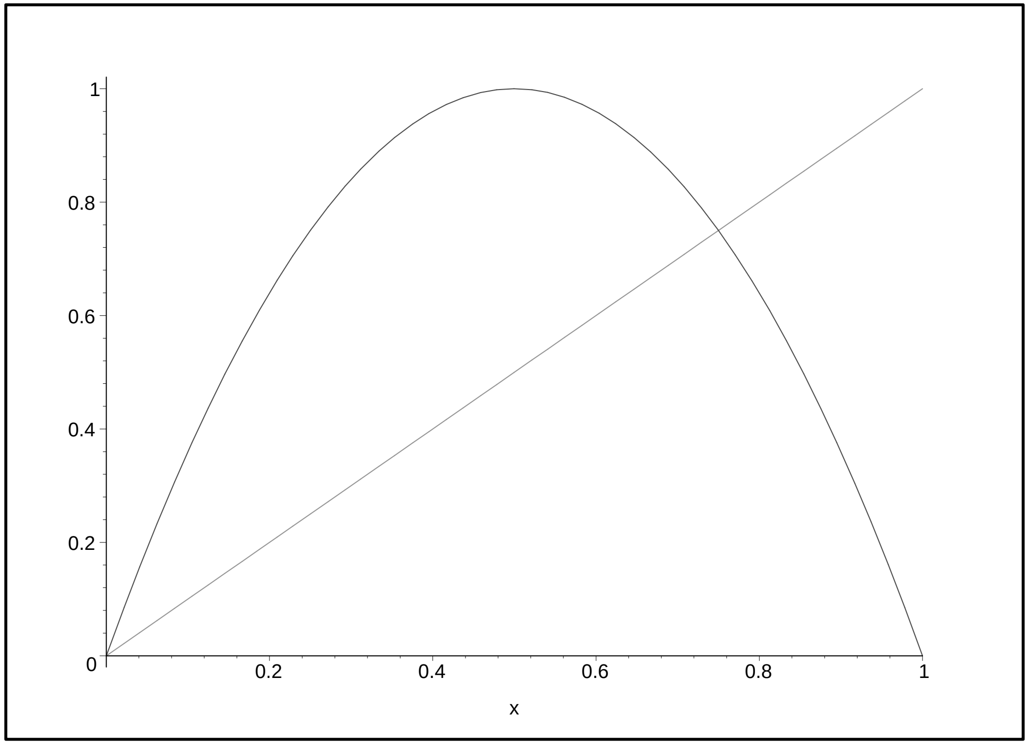

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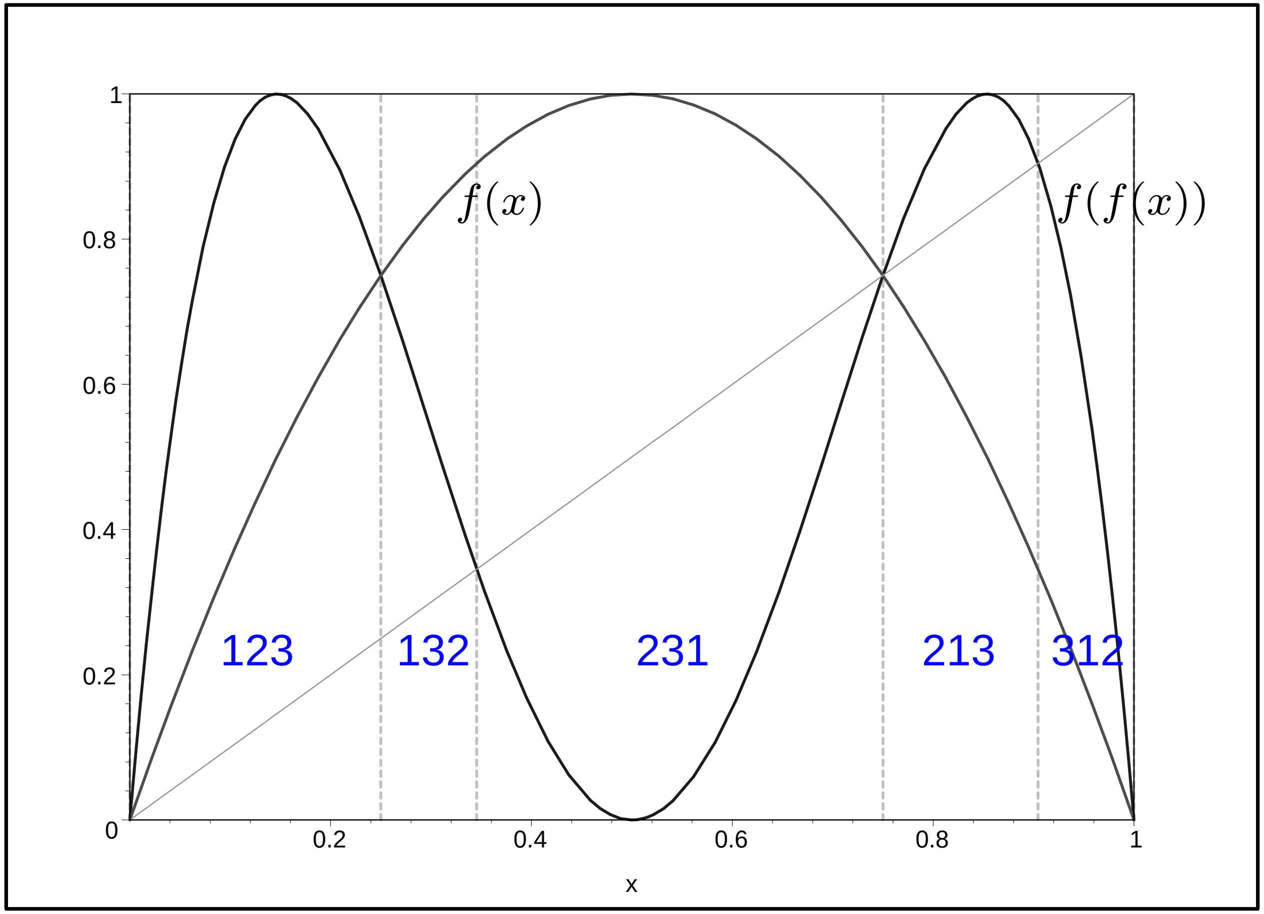
$x = 0.95$ defines the order pattern 312

How about the pattern 321?

Forbidden patterns

The pattern 321 does not appear for any x.



We say that 321 is a *forbidden pattern* of f.

Notation

$I \subset \mathbb{R}$ closed interval, $f : I \to I$, $x \in I$.

We say that x *defines the order pattern* $\pi \in \mathcal{S}_k$ if

$$(x, f(x), f(f(x)), \ldots, f^{(k-1)}(x)) \quad \sim \quad (\pi_1, \pi_2, \ldots, \pi_k),$$

where $(a_1, \ldots, a_k) \sim (b_1, \ldots, b_j)$ means that $a_i < a_j$ iff $b_i < b_j$.

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Given $\pi \in \mathcal{S}_k$, let

$$I_\pi = \{x \in I : x \text{ defines } \pi\}.$$

Let

$$\mathrm{Allow}_k(f) = \{\pi \in \mathcal{S}_k : I_\pi \neq \emptyset\}, \qquad \mathrm{Forb}_k(f) = \mathcal{S}_k \setminus \mathrm{Allow}_k(f).$$

$$\mathrm{Allow}(f) = \bigcup_{k \geq 1} \mathrm{Allow}_k(f), \qquad \mathrm{Forb}(f) = \bigcup_{k \geq 1} \mathrm{Forb}_k(f).$$

$\mathrm{Forb}(f)$ is the set of *forbidden patterns* of f.

Maps have forbidden patterns

Theorem. *If* $f : I \longrightarrow I$ *is a piecewise monotone map, then*

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This follows from a result of [Bandt, Keller, Pompe]:

$$|\mathrm{Allow}_k(f)| \propto e^{k\, h_{\mathrm{top}}(f)},$$

where $h_{\mathrm{top}}(f)$ is the topological entropy of f.

$\mathcal{P}$ partition of I into intervals on which f is strictly monotone.

$\mathcal{P}^{(n)}$ partition of I into all sets of the form
$P_1 \cap f^{-1}(P_2) \cap \cdots \cap f^{-(n-1)}(P_n)$ with $P_1, \ldots, P_n \in \mathcal{P}$.

Then the topological entropy of f is

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It follows that

$$|\text{Allow}_k(f)| \propto e^{k\, h_{\text{top}}(f)} << k! = |\mathcal{S}_k|,$$

so f has forbidden patterns.

Comparison with random sequences

Consider a sequence $x_1, x_2, \ldots, x_N$ produced by a black box, with $0 \leq x_i \leq 1$.

- If the sequence is of the form $x_{i+1} = f(x_i)$, for some piecewise monotone map f, then it must have missing patterns (if N large enough).

  For example, the pattern 321 is missing from

  $$.6416,\ .9198,\ .2951,\ .8320,\ .5590,\ .9861,\ .0550,\ .2078,\ .6584,\ .8996,\ .3612,\ .9230,\ .2844,\ .8141,\ .6054, \ldots.$$

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  Besides, the number of missing patterns of length k is at least $k! - C^k$, for some constant C.

- On the other hand, if the sequence was generated by N i.i.d. random variables, then the probability that any fixed pattern π is missing goes to 0 exponentially as N grows.

Consecutive patterns in permutations

$\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n, \quad \pi_1\pi_2\cdots\pi_k \in \mathcal{S}_k$

Definition. σ *contains* π as a consecutive pattern if there exists i such that

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$$\mathrm{Cont}_n(\pi) \ = \ \{\sigma \in \mathcal{S}_n \ : \ \sigma \text{ contains } \pi \text{ as a consecutive pattern}\}$$

$$\mathrm{Av}_n(\pi) \ = \ \{\sigma \in \mathcal{S}_n \ : \ \sigma \text{ avoids } \pi \text{ as a consecutive pattern}\}$$

$$\mathrm{Av}(\pi) = \bigcup_{n\geq 1} \mathrm{Av}_n(\pi)$$

Enumeration of permutations avoiding consecutive patterns

E., Noy:

- Formulas for the generating functions for the number of permutations avoiding a pattern of the form

$$\pi = 12 \cdots k \qquad \text{or}$$

$$\pi = 12 \cdots (a-1)a \underbrace{\cdots\cdots\cdots\cdots\cdots\cdots\cdots}_{\text{perm. of } \{a+2, a+3, \dots, k\}} (a+1).$$

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- For any $\pi \in \mathcal{S}_k$ with $k \geq 3$, there exist constants $0 < c, d < 1$ such that

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Kitaev:

- Formulas for the number of permutations avoiding multiple consecutive patterns of length 3.

$\mathrm{Allow}(f)$ *is closed under consecutive pattern containment*

$$\left.\begin{array}{c} \sigma \in \mathrm{Allow}(f) \\ \sigma \text{ contains } \tau \text{ as a consecutive pattern} \end{array}\right\} \quad \Rightarrow \quad \tau \in \mathrm{Allow}(f).$$

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Equivalently,

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Equivalently,

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We are interested in the minimal elements of $\mathrm{Forb}(f)$, i.e., those not containing any smaller pattern in $\mathrm{Forb}(f)$.

Root forbidden patterns

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Example: For $f(x) = 4x(1-x)$,

$\mathrm{Root}_2(f) = \emptyset$

$\mathrm{Root}_3(f) = \{\mathbf{321}\}$

$\mathrm{Root}_4(f) = \{\mathbf{1423}, \mathbf{2134}, \mathbf{2143}, \mathbf{3142}, \mathbf{4231}\}$

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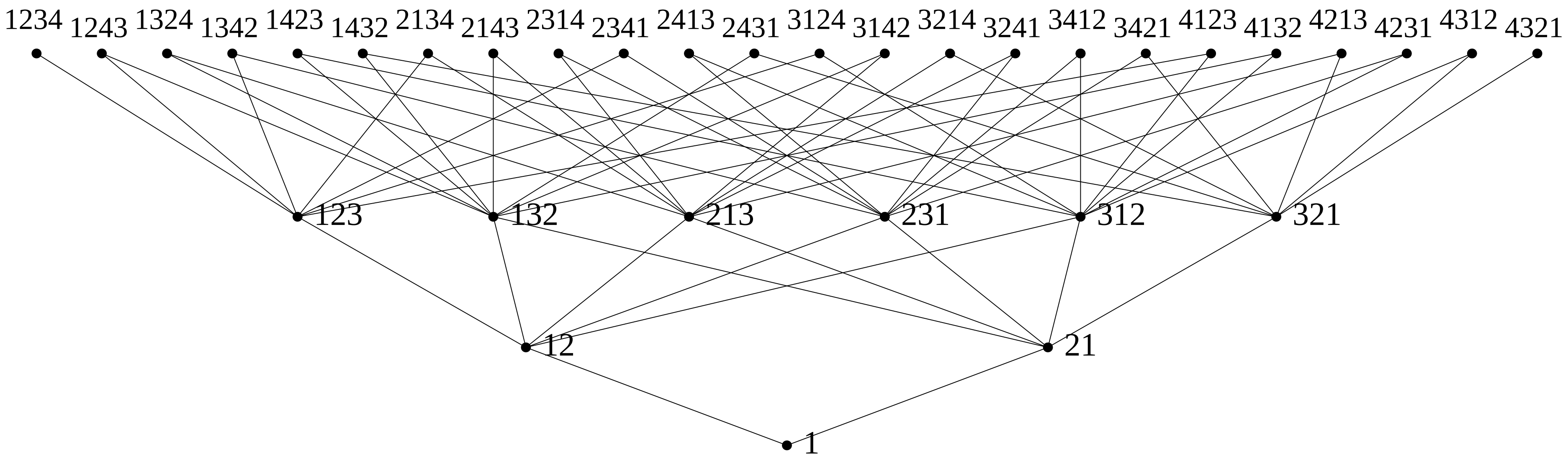
$\mathrm{Forb}_4(f) =$
$\{\mathbf{1423}, 1\underline{432}, \mathbf{2134}, \mathbf{2143}, 2\underline{431}, \mathbf{3142}, \underline{321}4, 3\underline{421}, \underline{421}3, \mathbf{4231}, \underline{431}2, \underline{4321}\}$

Poset of permutations under consec. pattern containment

We can consider the infinite poset of all permutations where

$\pi \leq \sigma \quad \Leftrightarrow \quad \sigma$ contains π as a consecutive pattern.



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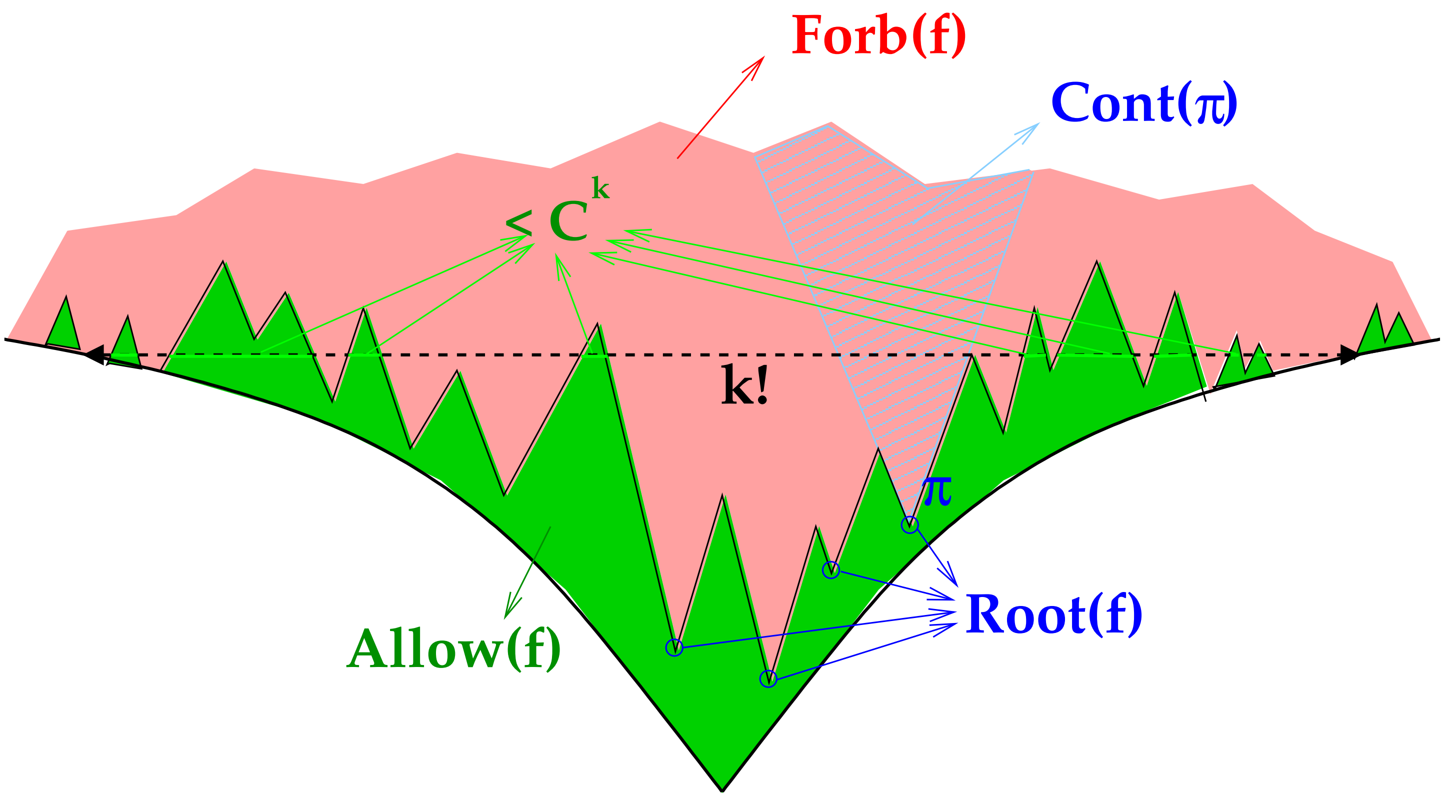
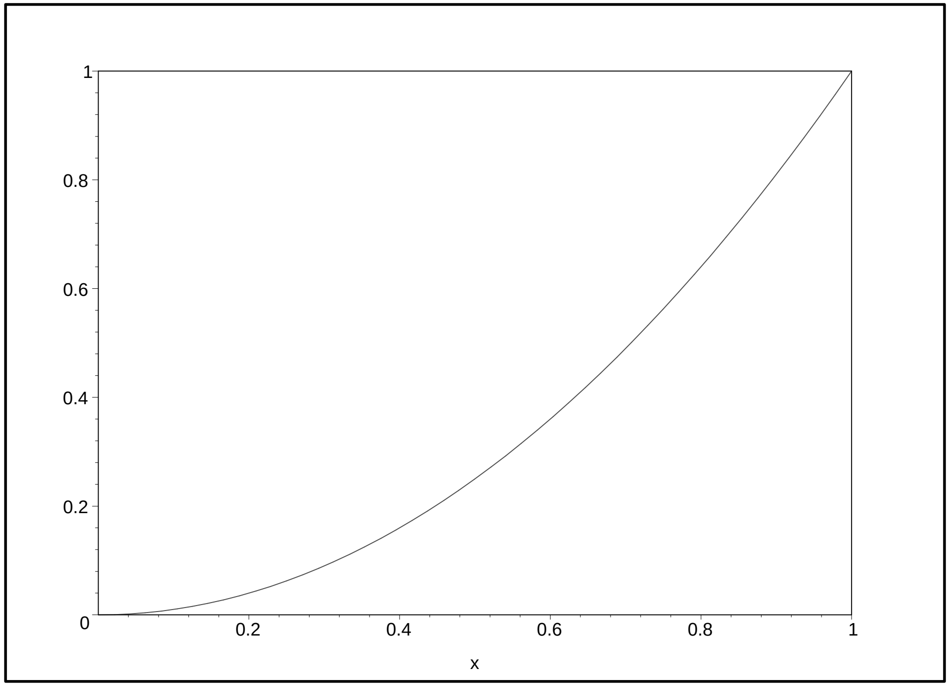


More examples

$$g_1(x) = x^2$$



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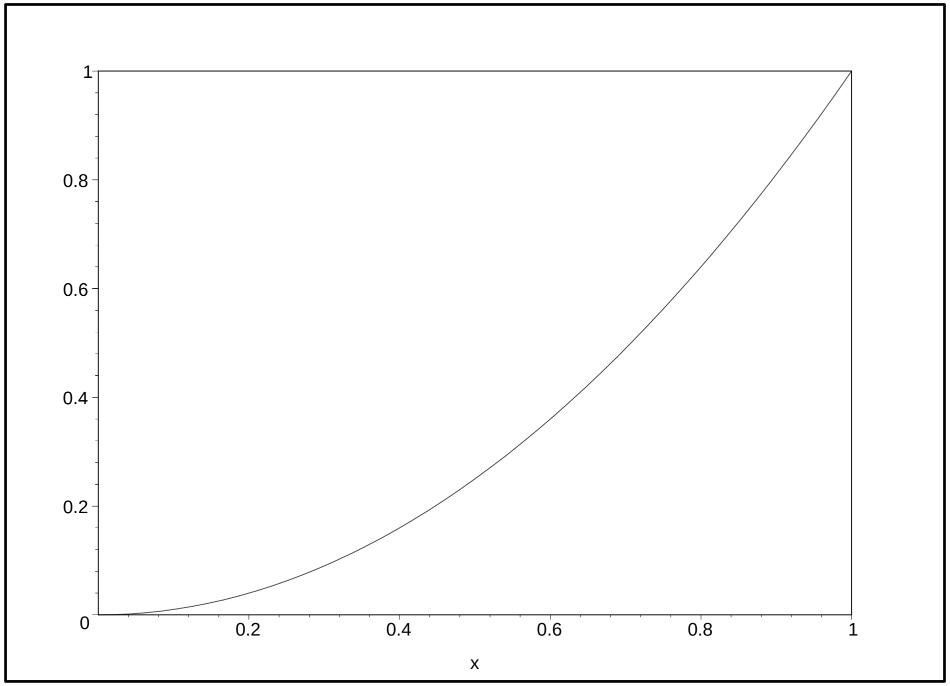
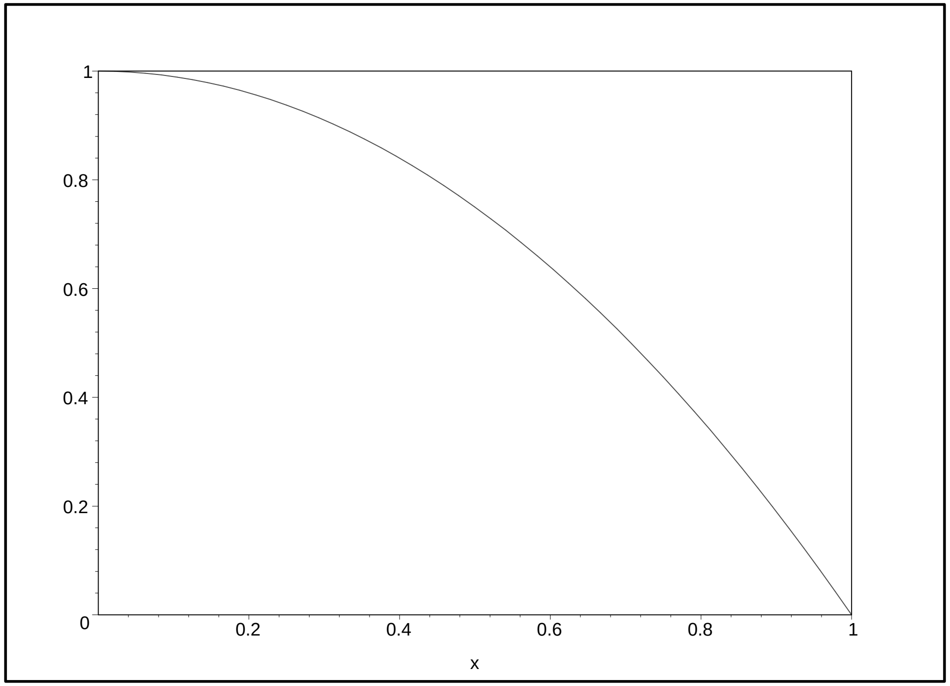


$\mathrm{Root}(g_1) = \{\mathbf{12}\}$

$\mathrm{Allow}_n(g_1) = \mathrm{Av}_n(\mathbf{12}) = \{n \ldots 21\}$

More examples

$$g_2(x) = 1 - x^2$$



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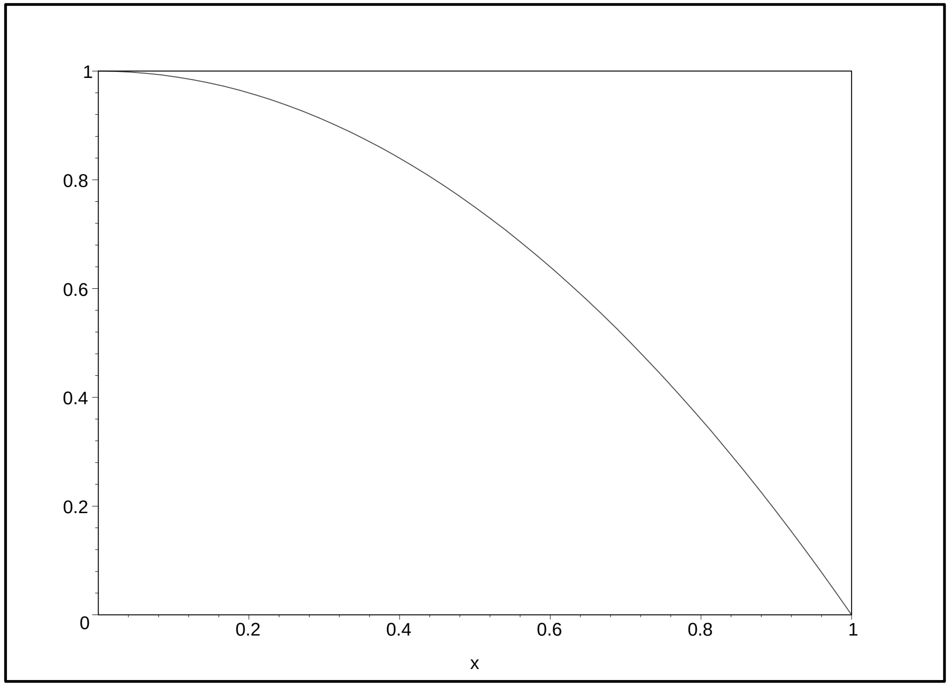

$$\mathrm{Root}(g_2) = \{\mathbf{123}, \mathbf{132}, \mathbf{312}, \mathbf{321}\}$$

$$\mathrm{Allow}_3(g_2) = \{213, 231\}$$

$$\mathrm{Allow}_4(g_2) = \{3241, 2314\}$$

$$\mathrm{Allow}_5(g_2) = \{32415, 34251\}$$

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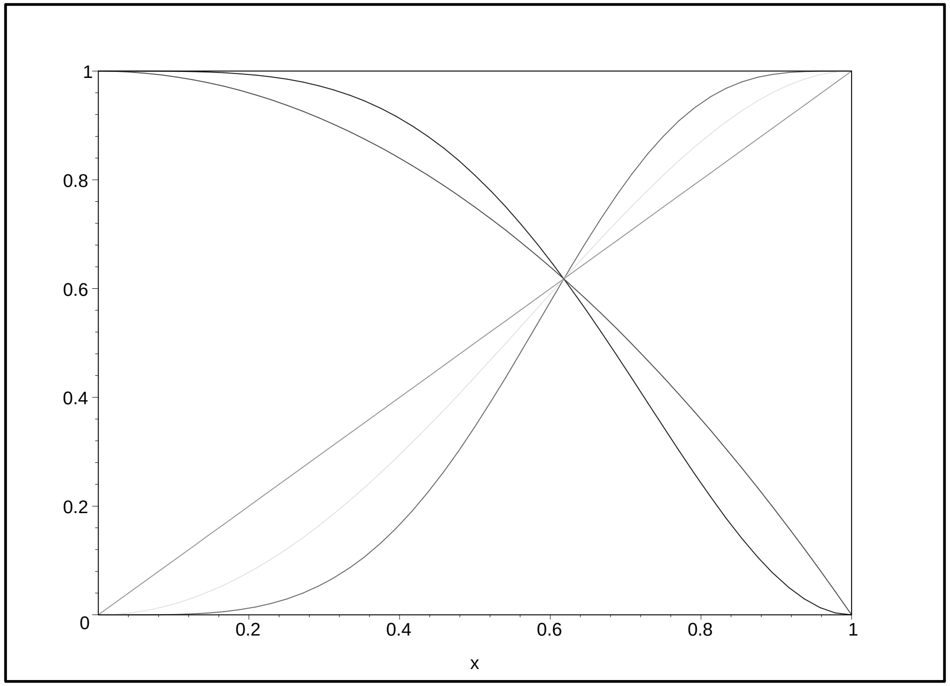

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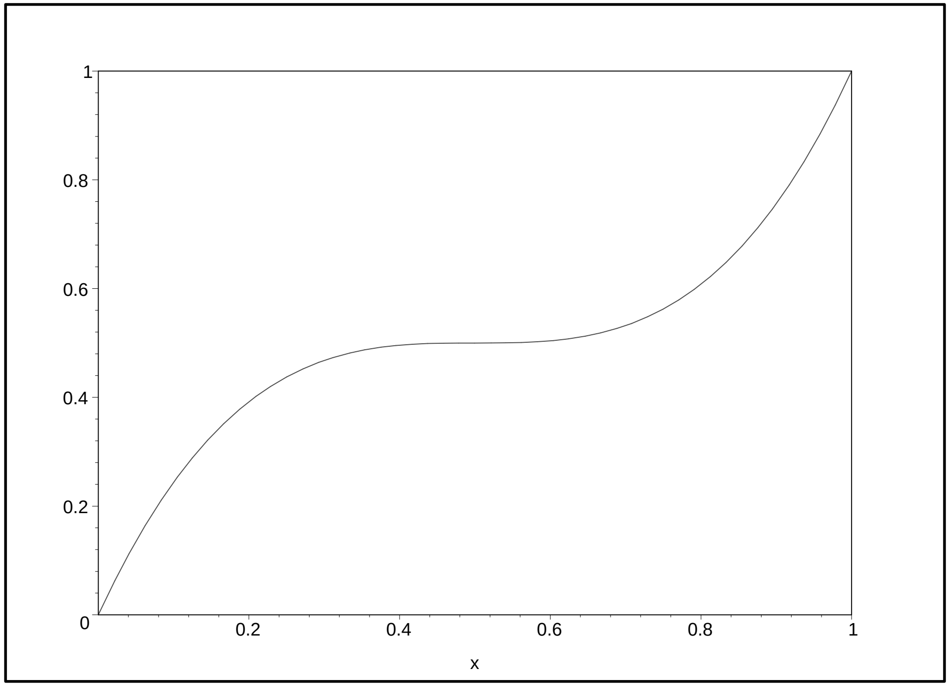
$$g_3(x) = 4x^3 - 6x^2 + 3x$$



More examples

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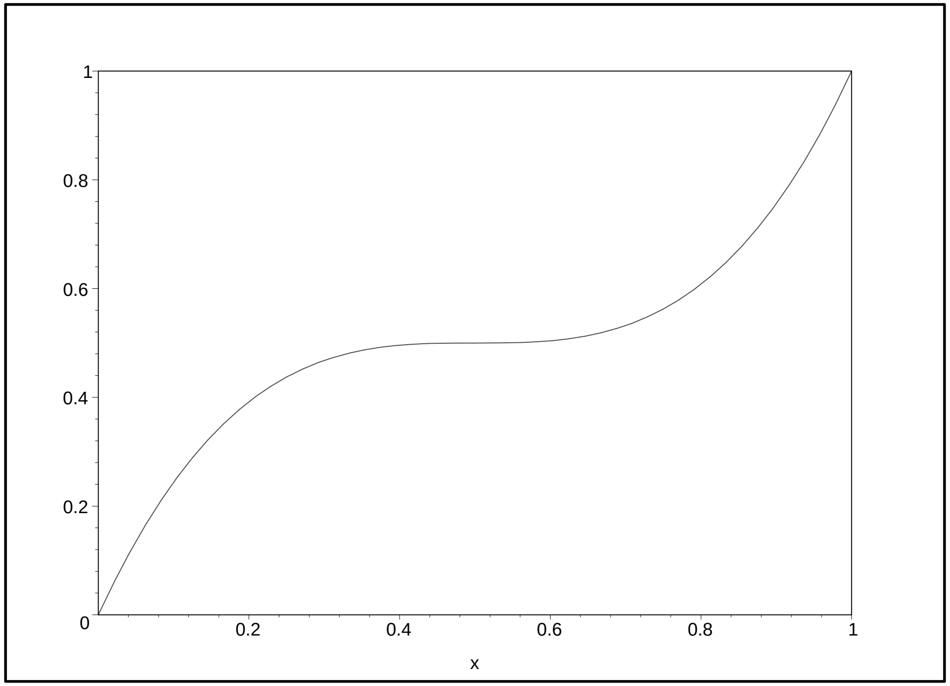

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$$\text{Allow}_n(g_3) = \{12\ldots n, n\ldots 21\}$$

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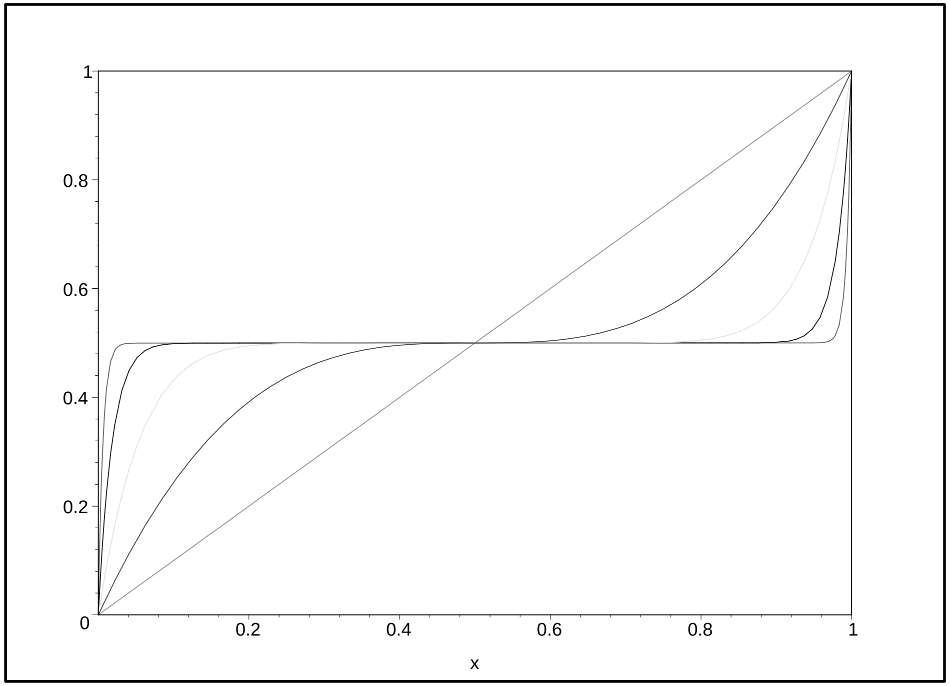

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Let

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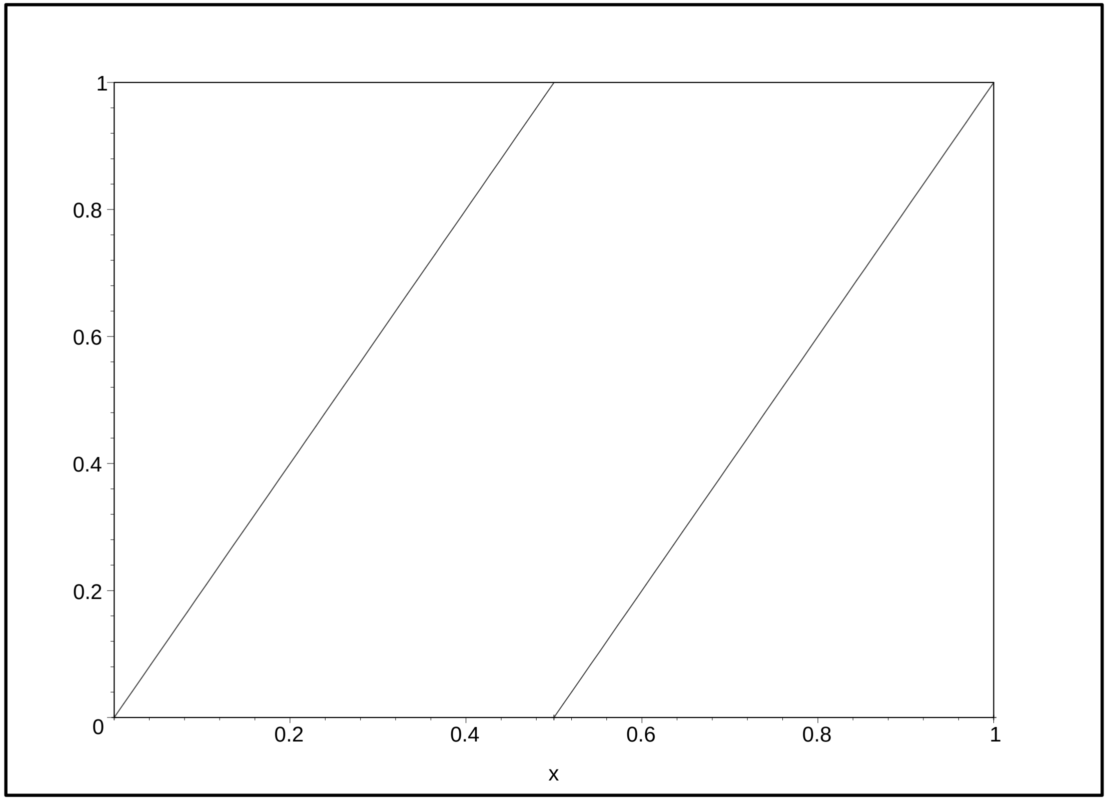
For $N \geq 2$, $\tilde{h}_N$ is called the *(one-sided) shift* on N symbols, and

$$h_N : x \mapsto Nx \mod 1$$

is called the *sawtooth map*.

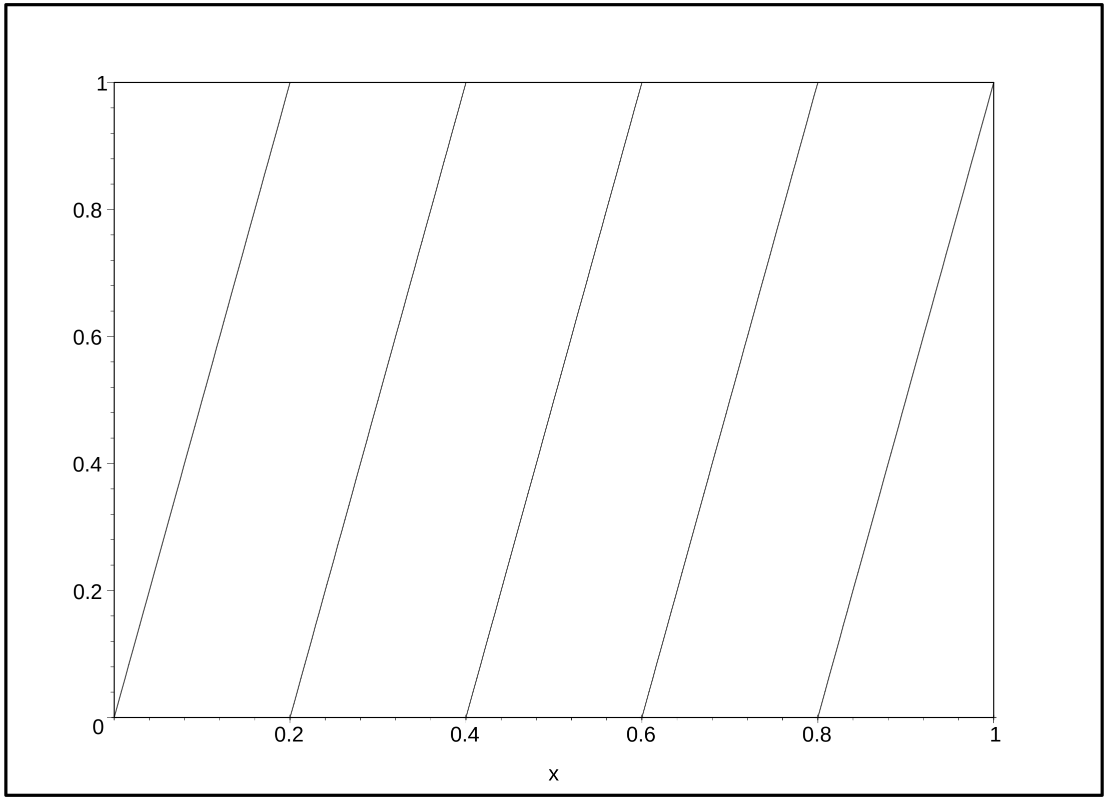
Sawtooth map

$$h_2 : [0,1] \to [0,1]$$



Sawtooth map

$$h_5 : [0,1] \to [0,1]$$



Order patterns in $h_N : [0,1] \to [0,1]$ $\leftrightsquigarrow$ Order patterns in $\tilde{h}_N : \{0,1,\dots,N-1\}^* \to \{0,1,\dots,N-1\}^*$ using the lexicographic order

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Example. For $\tilde{h}_3$ and $k=7$, the point $x = (2,1,0,2,2,1,2,2,1,0,\ldots)$ defines the pattern 4217536.

$$(2,1,0,2,2,1,2,2,1,0,\ldots) \overset{\tilde{h}_3}{\mapsto} (1,0,2,2,1,2,2,1,0,\ldots) \overset{\tilde{h}_3}{\mapsto} (0,2,2,1,2,2,1,0,\ldots) \overset{\tilde{h}_3}{\mapsto}$$
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Theorem.

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Forbidden patterns in shift systems

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Conjecture. *For all* $N \geq 2$, h_N *has exactly 6 forbidden patterns of length* $N+2$.

The condition of piecewise monotonicity is essential:

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- Decompose $[0,1]$ into infinitely many intervals, e.g.,

$$[0,1] = \bigcup_{N \geq 2} I_N, \quad \text{where} \quad I_N = \left[\frac{1}{2^{N-1}}, \frac{1}{2^{N-2}}\right).$$

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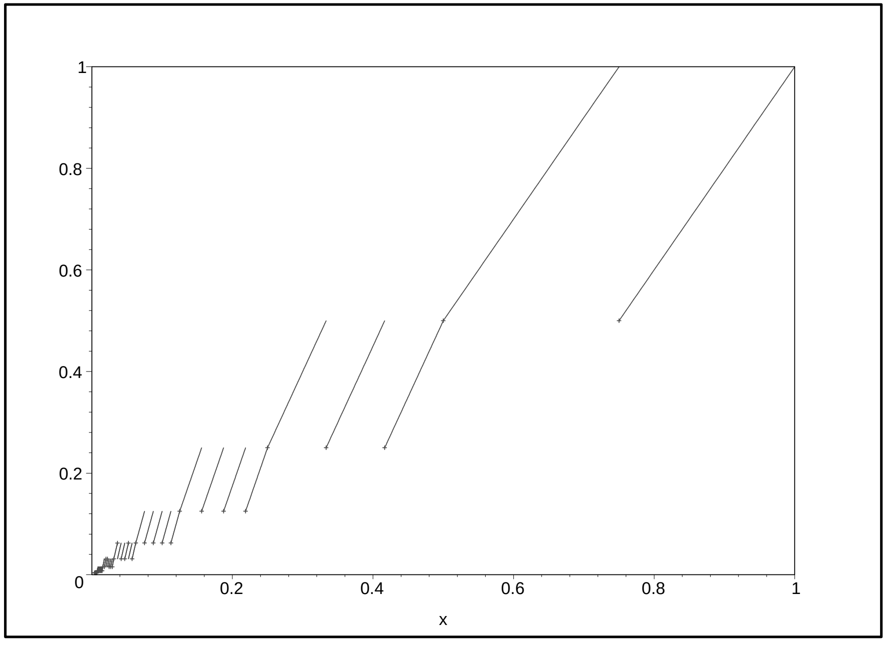
- Define on each I_N a properly scaled version of h_N from I_N to I_N.



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- Characterize the maps f for which $\mathrm{Root}(f)$ is finite.

- Is there an efficient algorithm to find $\mathrm{Root}(f)$, given f in some suitable class?

  How about to find the length of the smallest forbidden pattern?

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Note: Σ must be such that $\mathrm{Av}_n(\Sigma) < C^n$ for some constant C, since $\mathrm{Av}_n(\Sigma) = \mathrm{Allow}_n(f)$.

For example, if $\Sigma = \{\pi\}$, where π has length at least 3, then there is no f such that $\mathrm{Allow}(f) = \mathrm{Av}(\pi)$, because $\mathrm{Av}_n(\pi) > \lambda^n n!$ for some $0 < \lambda < 1$.

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- What else can we say about the structure or the asymptotic growth of $\mathrm{Allow}(f)$ or $\mathrm{Forb}(f)$?

Discrete Mathematics Day

Saturday, October 27, 2007

Dartmouth College

Hanover, NH

`http://math.dartmouth.edu/∼dmd`

Speakers:

- **Jim Propp**, University of Massachusetts at Lowell
- **Vera Sos**, Mathematical Institute of the Hungarian Academy of Sciences, Budapest
- **Mikkel Thorup**, AT&T Labs Research
- **Lauren Williams**, Harvard University
- **Josephine Yu**, MIT