The Structure of the Consecutive Pattern Poset

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Outline

- Classical and consecutive patterns
- The consecutive pattern poset
- Results
- Open problems

Classical patterns

Definition. An occurrence of a permutation σ as a pattern in a permutation τ is a subsequence of τ whose letters are in the same relative order as those in σ.

Examples.

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This is NOT the definition that we will focus on.

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Work in the area by Aldred, Amigó, Atkinson, Bandt, Baxter, Bernini, Bóna, Dotsenko, Duane, Dwyer, Ehrenborg, Ferrari, Keller, Kennel, Khoroshkin, Kitaev, Liese, Liu, Mansour, McCaughan, Mendes, Nakamura, Noy, Perarnau, Perry, Pompe, Pudwell, Rawlings, Remmel, Sagan, Shapiro, Steingrímsson, Warlimont, Willenbring, Zeilberger . . .

A sample of known results on consecutive patterns

For a fixed pattern σ, let

$$P_\sigma(u,z) = \sum_{n\geq 0} \sum_{\pi\in S_n} u^{\#\{\text{occurrences of } \sigma \text{ in } \pi\}} \frac{z^n}{n!},$$

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$$P_{132}(u,z) = \left(1 - \int_0^z e^{(u-1)t^2/2}\, dt\right)^{-1}$$

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Also for σ monotone; σ non-overlapping with $\sigma_1 = 1$; $\sigma = 1324$; etc.

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2. Classification according to *consecutive Wilf-equivalence*: Let

$$\sigma \sim \tau \quad \text{if} \quad P_\sigma(0,z) = P_\tau(0,z),$$
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Theorem [E. '13] For every $\sigma \in S_m$ there exists n_0 such that

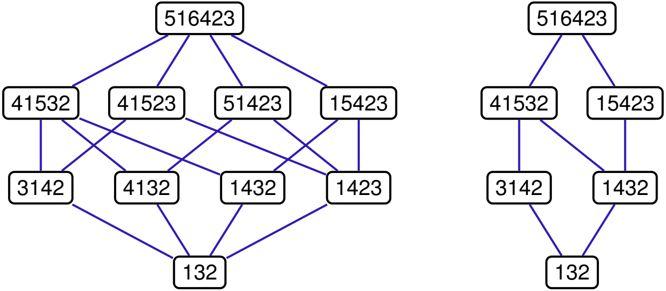
$$\alpha_n(123\dots(m-2)m(m-1)) \leq \alpha_n(\sigma) \leq \alpha_n(12\dots m)$$

for all $n \geq n_0$.

Pattern posets

Order permutations by pattern containment:
$\sigma \leq \tau$ if σ occurs as a pattern in τ.

Classical patterns **Consecutive patterns**



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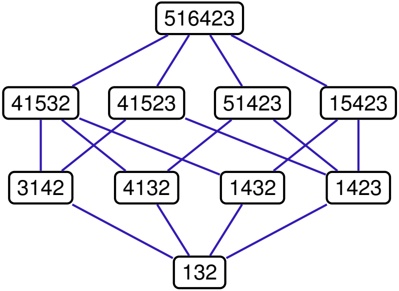

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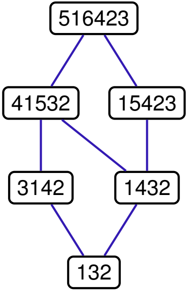
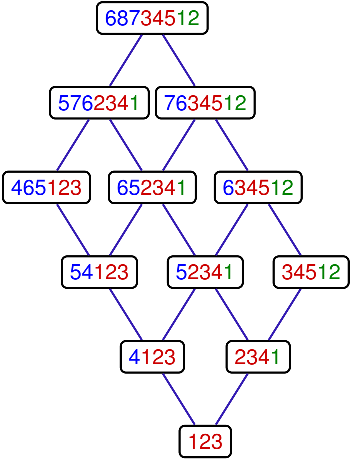


The consecutive pattern poset is more manageable:

- Every permutation covers at most two others.
- The Möbius function is known [Bernini–Ferrari–Steingrímsson, Sagan–Willenbring '11], unlike in the clasical case.

Pattern posets

- In the consecutive pattern poset, when σ occurs just once in τ, $[\sigma, \tau]$ is a product of two chains [BFS11].



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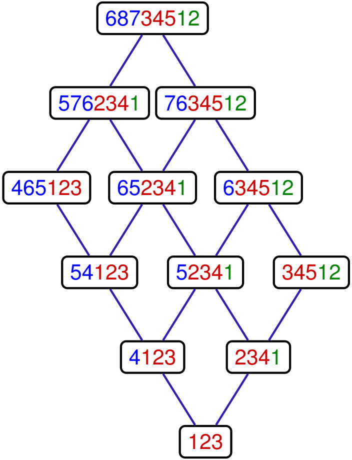
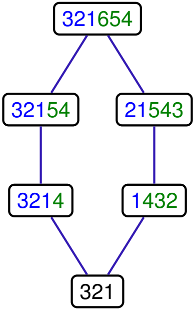


No analogue for classical pattern poset.

Main questions

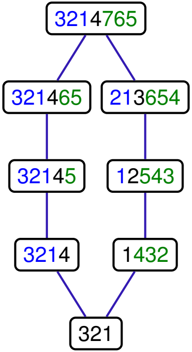
Unless otherwise specified: consecutive pattern poset.



1. Which open intervals are disconnected?
2. Which intervals are shellable?
3. Which intervals are rank-unimodal?
4. Which intervals are (strongly) Sperner?
5. Which intervals have Möbius function equal to 0?

1. Which open intervals are disconnected?

Definition. For $\sigma < \tau$, we say that σ **straddles** τ if σ is both a prefix and suffix of τ and has no other occurrences in τ.



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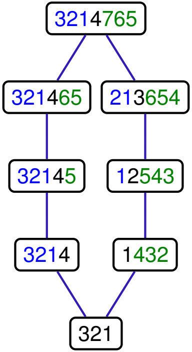

Theorem

For $\sigma < \tau$ with $|\tau| - |\sigma| \geq 3$, the open interval (σ, τ) is disconnected if and only if σ straddles τ.
In this case, (σ, τ) consists of two disjoint chains.

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Some combinatorial topology...

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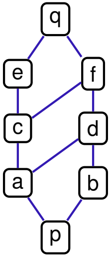

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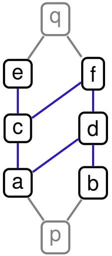

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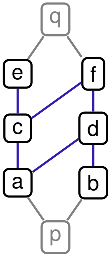

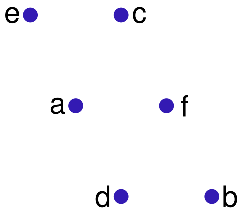

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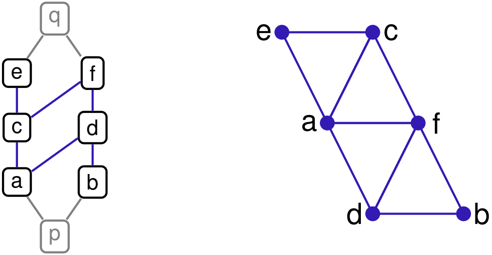

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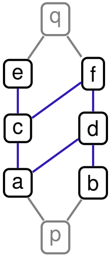

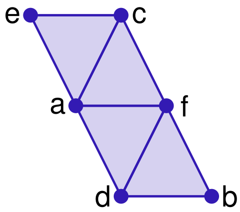

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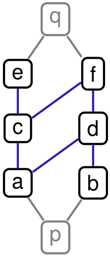

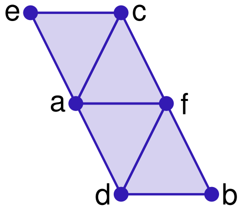

Definition. A pure d-dimensional complex is shellable if its facets can be ordered $F_1, F_2, \ldots, F_n$ such that, for all $2 \leq i \leq n$,

$$F_i \cap (F_1 \cup F_2 \cup \cdots \cup F_{i-1})$$

is pure and $(d-1)$-dimensional.

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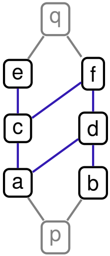

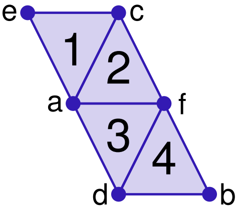

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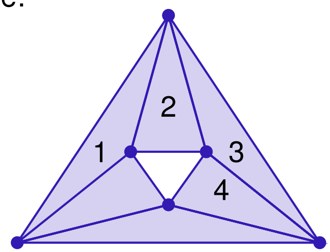
Shellability

Non-shellable example:



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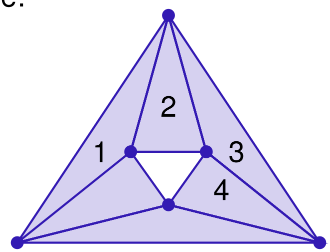
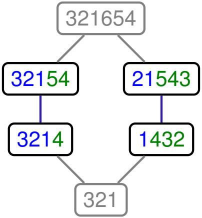


Why we care about shellability:

- Shellable $\Rightarrow$ contractible, or homotopic to a wedge of spheres in the top dimension.
- Combinatorial tools for showing shellability of $\Delta(P)$: EL-shellability, CL-shellability, etc.

Disconnected and non-shellable

Easy non-shellable example: If (σ, τ) disconnected with $|\tau| - |\sigma| \geq 3$, then $\Delta(\sigma, \tau)$ is not shellable.



We call this a non-trivial disconnected interval.

If $[\sigma, \tau]$ contains a non-trivial disconnected subinterval, then $[\sigma, \tau]$ is not shellable.

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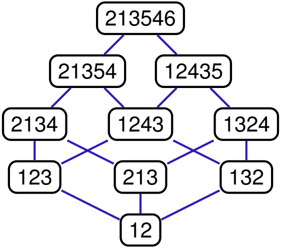
Theorem

Fix σ, and let $\tau \in S_n$ be uniformly random. Then

$$\lim_{n\to\infty} (\textit{Probability that } [\sigma, \tau] \textit{ is shellable}) = 0.$$

3. Which intervals are rank-unimodal?



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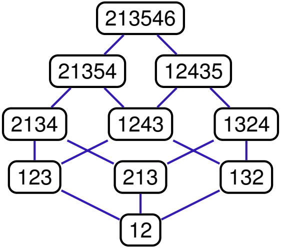

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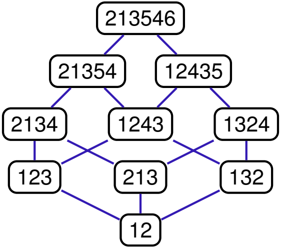

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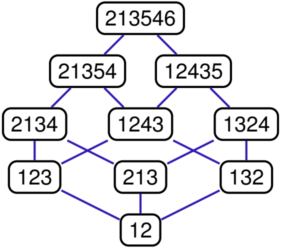

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- Top part is grid-like.
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Conjecture [McNamara–Steingrímsson '15]

Every interval $[\sigma, \tau]$ in the classical pattern poset is rank-unimodal.
(True for intervals of rank ≤ 8.)

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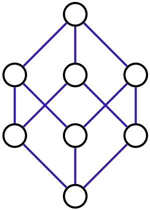
Example.

Sperner: 

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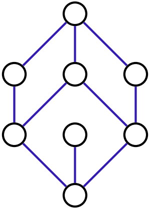

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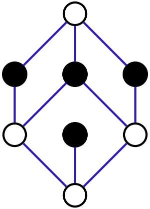
Not Sperner: 

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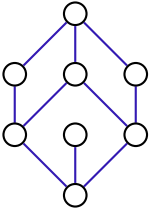
Not Sperner: 

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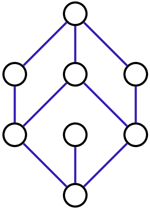

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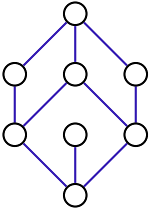

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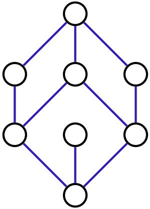

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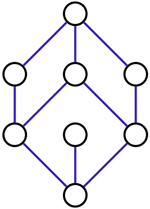

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The proof uses a result of Griggs, plus the injections from our rank-unimodality proof.

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Interior $i(\tau)$: the permutation pattern obtained by deleting first and last element of τ.

Exterior $x(\tau)$: the longest proper prefix that is also a suffix (as a pattern).

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$\tau = 21435,\ i(\tau) = 132,\ x(\tau) = 213$
$\tau = 123456$ (monotone), $x(\tau) = 12345$
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Theorem [BFS, SW '11]. For $\sigma \leq \tau$,

$$\mu(\sigma,\tau) = \begin{cases} \mu(\sigma, x(\tau)) & \text{if } |\tau| - |\sigma| > 2 \text{ and } \sigma \leq x(\tau) \not\leq i(\tau), \\ 1 & \text{if } |\tau| - |\sigma| = 2, \tau \text{ is not monotone,} \\ & \text{and } \sigma \in \{i(\tau), x(\tau)\}, \\ (-1)^{|\tau|-|\sigma|} & \text{if } |\tau| - |\sigma| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

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Crucial role played by $x(\tau)$.

Length of the exterior

Number of permutations $\tau \in S_n$ with $|x(\tau)| = k$:

$n \backslash k$	1	2	3	4	5	6	7	8	9
2	2								
3	4	2							
4	12	10	2						
5	48	58	12	2					
6	280	306	118	14	2				
7	1864	2186	822	150	16	2			
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