Counting lattice paths by the number of crossings and major index

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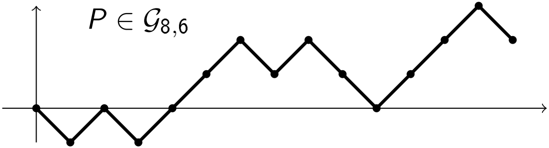
Lattice Paths, Combinatorics and Interactions - June 2021

I. Paths crossing a line

Lattice paths and major index

Let $\mathcal{G}_{a,b}$ be the set of lattice paths in $\mathbb{Z}^2$ with a steps $U = (1,1)$ and b steps $D = (1,-1)$, starting at the origin.

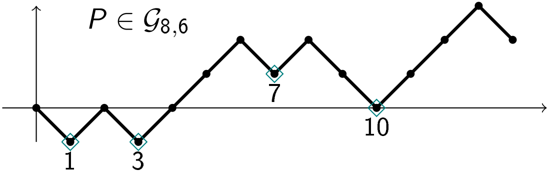


Lattice paths and major index

Let $\mathcal{G}_{a,b}$ be the set of lattice paths in $\mathbb{Z}^2$ with a steps $U = (1,1)$ and b steps $D = (1,-1)$, starting at the origin.

Encoding paths $P \in \mathcal{G}_{a,b}$ as binary words via $U \mapsto 0$, $D \mapsto 1$, we have these definitions:

- a descent of P is a valley, i.e., a corner DU,
- the major index, $\mathrm{maj}(P)$, is the sum of the x-coordinates of the valleys



$$\mathrm{maj}(P) = 1 + 3 + 7 + 10 = 21$$

Lattice paths and major index

Lemma (MacMahon)

$$\sum_{P \in \mathcal{G}_{a,b}} q^{\operatorname{maj}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

is a *q-binomial coefficient*.

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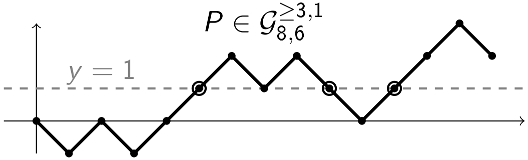
For $\ell \in \mathbb{Z}$ and $r \geq 0$, let $\mathcal{G}_{a,b}^{\geq r,\ell}$ be the set of paths in $\mathcal{G}_{a,b}$ that cross the line $y = \ell$ at least r times.



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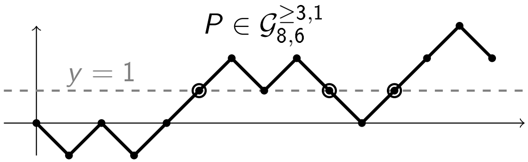

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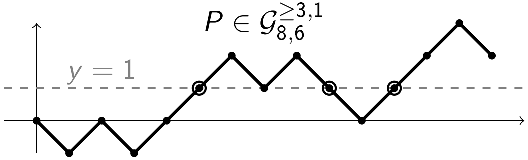

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We are interested in the polynomials

$$G_{a,b}^{\geq r,\ell}(q) = \sum_{P \in \mathcal{G}_{a,b}^{\geq r,\ell}} q^{\mathrm{maj}(P)}.$$

Counting paths crossing the x-axis

Consider first the case where $\ell = 0$.

Theorem

For any $a, b, r \geq 0$,

$$G^{\geq r,0}_{a,b}(q) = \begin{cases} q^{\binom{r+1}{2}} \begin{bmatrix} a+b \\ a+r \end{bmatrix}_q & \text{if } a > b, \\ (1+q^a)q^{\binom{r+1}{2}} \begin{bmatrix} 2a-1 \\ a+r \end{bmatrix}_q & \text{if } a = b, \\ q^{\binom{r}{2}} \begin{bmatrix} a+b \\ a-r \end{bmatrix}_q & \text{if } a < b. \end{cases}$$

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Our proof is bijective.

Connections to the literature

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- The theorem has applications to the enumeration of partitions λ with certain restrictions on the ranks $\lambda_i - \lambda'_i$, studied by Corteel–E.–Savage '21+.

Counting paths crossing a horizontal line

Theorem

Let $a, b, m \geq 0$, *and let* $\ell \in \mathbb{Z} \setminus \{0\}$. *If* $0 < \ell < a - b$, *then*

$$G_{a,b}^{\geq 2m+1,\ell}(q) = G_{a,b}^{\geq 2m,\ell}(q) = q^{m(2m+1+\ell)} \begin{bmatrix} a+b \\ a+2m \end{bmatrix}_q .$$

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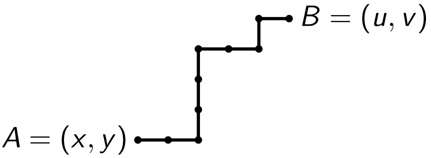
II. Pairs of paths crossing each other

Paths with north and east steps

For $A, B \in \mathbb{Z}^2$, let $\mathcal{P}_{A\to B}$ be the set of lattice paths from A to B with steps $N = (0,1)$ and $E = (1,0)$.



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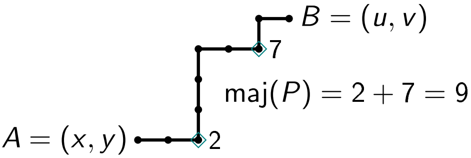
Descents of $P \in \mathcal{P}_{A\to B}$ are corners EN, and $\text{maj}(P)$ is the sum of the positions of the valleys, where the position is determined by numbering the vertices of P starting from 0.



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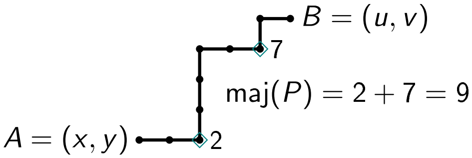


If $A = (x, y)$ and $B = (u, v)$, MacMahon's formula gives

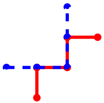
$$\sum_{P\in\mathcal{P}_{A\to B}} q^{\text{maj}(P)} = \begin{bmatrix} u-x+v-y \\ u-x \end{bmatrix}_q .$$

Crossings of two paths

A crossing of two paths P and Q is a common vertex C such that:

- P and Q disagree in the step arriving at C;
- at the first step after C where P and Q disagree, each path has the same type of step (N or E) as it had when arriving at C.

crossings



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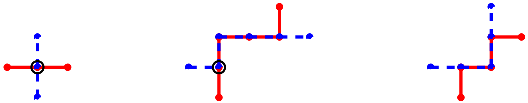

crossings   not a crossing

$$\mathcal{P}^{\geq r}_{A_1 \to B_\circ, A_2 \to B_\bullet} = \{(P, Q) : P \in \mathcal{P}_{A_1 \to B_\circ}, Q \in \mathcal{P}_{A_2 \to B_\bullet}, \\ P \text{ and } Q \text{ have } \geq r \text{ crossings}\}.$$

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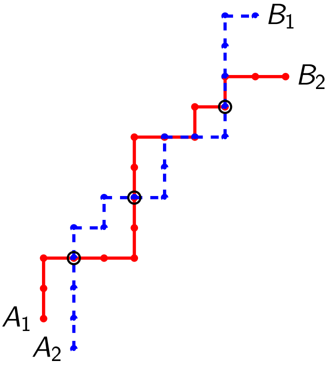
A pair in $\mathcal{P}^{\geq 3}_{A_1 \to B_2, A_2 \to B_1}$:



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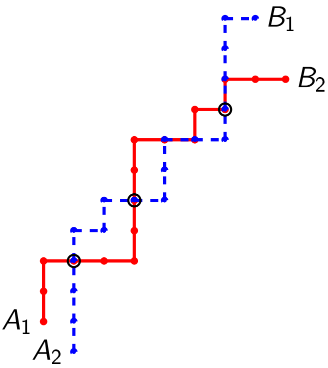

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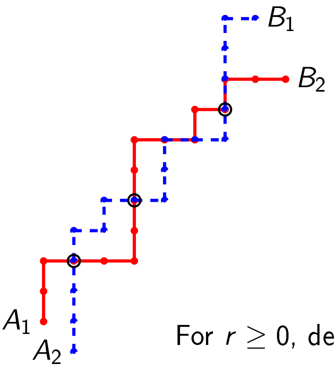

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For $r \geq 0$, define the polynomials

$$H^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}(q) = \sum_{(P,Q)\in\mathcal{P}^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}} q^{\mathrm{maj}(P)+\mathrm{maj}(Q)}.$$

Easy cases and notation

Let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $B_\circ = (u_\circ, v_\circ)$, $B_\bullet = (u_\bullet, v_\bullet)$.

For $r = 0$, we can choose the two paths independently, so

$$H^{\geq 0}_{A_1 \to B_\circ, A_2 \to B_\bullet}(q) = \begin{bmatrix} u_\circ - x_1 + v_\circ - y_1 \\ u_\circ - x_1 \end{bmatrix}_q \begin{bmatrix} u_\bullet - x_2 + v_\bullet - y_2 \\ u_\bullet - x_2 \end{bmatrix}_q .$$

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To give a general formula, first define

$$f_r(A_1, A_2, B_\circ, B_\bullet; q) := q^{r(r+x_2-x_1)} \begin{bmatrix} u_\circ - x_1 + v_\circ - y_1 \\ u_\circ - x_1 + r \end{bmatrix}_q \begin{bmatrix} u_\bullet - x_2 + v_\bullet - y_2 \\ u_\bullet - x_2 - r \end{bmatrix}_q .$$

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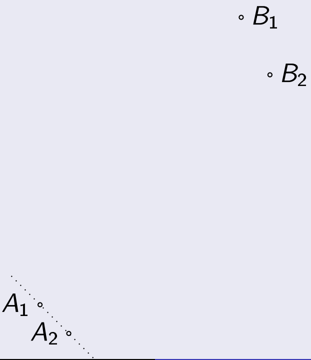
Write $A_1 \prec A_2$ to mean that A_1 is strictly northwest of A_2.

Counting pairs of paths by crossings

Theorem

Let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $B_1 = (u_1, v_1)$, $B_2 = (u_2, v_2)$, *where* $A_1 \prec A_2$ *and* $B_1 \prec B_2$, *and* $x_1 + y_1 = x_2 + y_2$.



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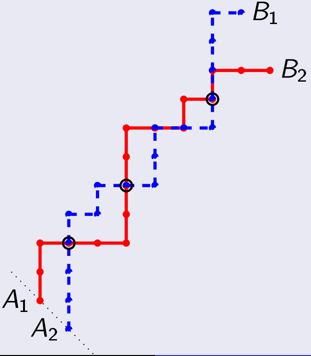

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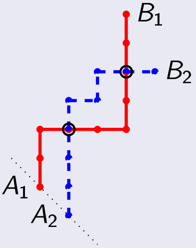

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Now let $A = (x, y)$ *and* $B = (u, v)$. *Then, for all* $r \geq 0$,

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Theorem

Let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $B_1 = (u_1, v_1)$, $B_2 = (u_2, v_2)$, *where* $A_1 \prec A_2$ *and* $B_1 \prec B_2$, *and* $x_1 + y_1 = x_2 + y_2$. *Then, for all* $m \geq 0$,

$$H^{\geq 2m+1}_{A_1 \to B_2, A_2 \to B_1}(q) = H^{\geq 2m}_{A_1 \to B_2, A_2 \to B_1}(q) = f_{2m}(A_1, A_2, B_2, B_1; q),$$

and for all $m \geq 1$,

$$H^{\geq 2m}_{A_1 \to B_1, A_2 \to B_2}(q) = H^{\geq 2m-1}_{A_1 \to B_1, A_2 \to B_2}(q) = f_{2m-1}(A_1, A_2, B_2, B_1; q).$$

Now let $A = (x, y)$ *and* $B = (u, v)$. *Then, for all* $r \geq 0$,

$$H^{\geq r}_{A \to B_1, A \to B_2}(q) = f_r(A, A, B_1, B_2; q),$$

$$H^{\geq r}_{A_1 \to B, A_2 \to B}(q) = f_r(A_1, A_2, B, B; q),$$

$$H^{\geq r}_{A \to B, A \to B}(q) = \begin{cases} f_0(A, A, B, B; q) & \text{if } r = 0, \\ 2\sum_{j \geq 1} (-1)^{j-1} f_{r+j}(A, A, B, B; q) & \text{if } r \geq 1. \end{cases}$$

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With the specialization $q = 1$ (which ignores maj), the theorem still holds when removing the requirement $x_1 + y_1 = x_2 + y_2$.

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However, this method does not prove the refinement by maj.

Our proof of the refined case is related to Krattenthaler's '95 refinement of the Gessel–Viennot determinant by maj. However, our bijections have simple descriptions in terms of paths.

III. Some bijections used in the proofs

The bijections $\bar{\tau}$ and $\bar{\sigma}$

Partition $\mathcal{P}_{A\to B} = \mathcal{P}^E_{A\to B} \cup \mathcal{P}^N_{A\to B}$ according to the last step of the path. Let $\mathbf{v} = (1, -1)$.

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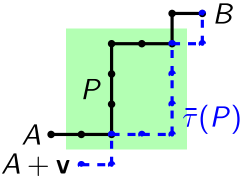
Define a bijection

$$\bar{\tau} : \mathcal{P}^E_{A\to B} \to \mathcal{P}^N_{A+\mathbf{v}\to B}$$

by placing the NE corners of $\bar{\tau}(P)$ at the coordinates of the EN corners of P:



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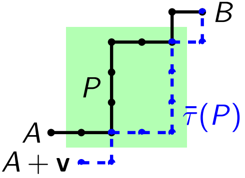


If $A = (x,y)$ and $B = (u,v)$, one can show that

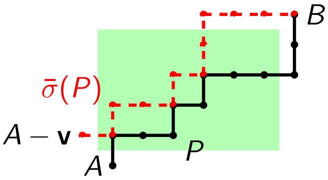
$$\mathrm{maj}(\bar{\tau}(P)) = \mathrm{maj}(P) + u - x - 1.$$

The bijections $\bar{\tau}$ and $\bar{\sigma}$

Similarly, define a bijection

$$\bar{\sigma} : \mathcal{P}^N_{A\to B} \to \mathcal{P}^E_{A-\mathbf{v}\to B}$$

by placing the EN corners of $\bar{\sigma}(P)$ at the coordinates of the NE corners of P:



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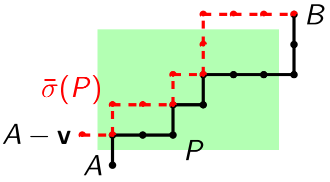
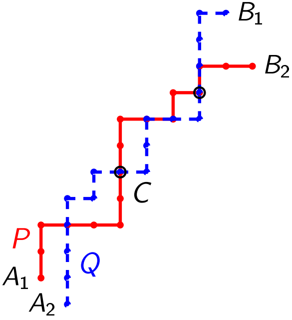


If $A = (x, y)$ and $B = (u, v)$, one can show that

$$\text{maj}(\bar{\sigma}(P)) = \text{maj}(P) - u + x.$$

A bijection for pairs of paths

Given $(P,Q) \in \mathcal{P}^{\geq r}_{A_1 \to B_\circ, A_2 \to B_\bullet}$, let C be the rth crossing from the right. Suppose that P arrives to C with an N, and Q with an E.



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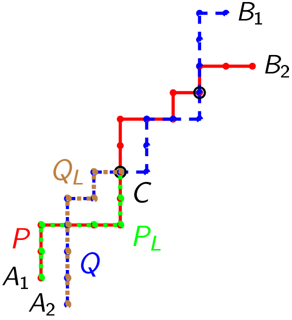
Splitting the paths at C, write $P = P_L P_R$ and $Q = Q_L Q_R$.



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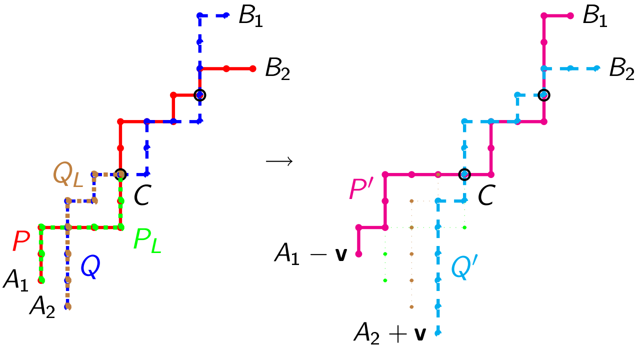
Splitting the paths at C, write $P = P_L P_R$ and $Q = Q_L Q_R$. Let

$$P' = \bar{\sigma}(P_L) Q_R \in \mathcal{P}_{A_1 - \mathbf{v} \to B_\bullet} \quad \text{and} \quad Q' = \bar{\tau}(Q_L) P_R \in \mathcal{P}_{A_2 + \mathbf{v} \to B_\circ}.$$



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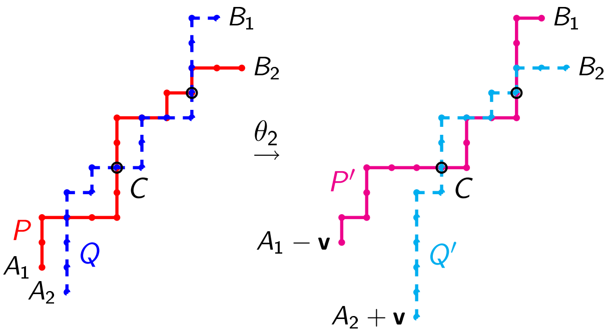

With the right setup, the map $(P, Q) \mapsto (P', Q')$ is a bijection, which we denote by θ_r.

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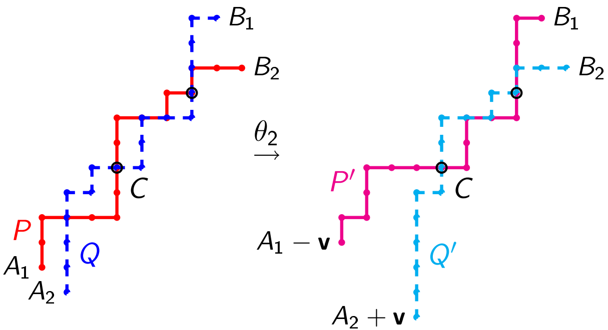

With the right setup, the map $(P, Q) \mapsto (P', Q')$ is a bijection, which we denote by θ_r.

If $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$, one can show that

$$\text{maj}(P') + \text{maj}(Q') = \text{maj}(P) + \text{maj}(Q) - (x_2 - x_1 + 1).$$

Composing bijections

To prove our theorem about pairs of paths, we use compositions such as $\theta_1 \circ \theta_2 \circ \cdots \circ \theta_r$, which decreases maj by $r(r + x_2 - x_1)$.

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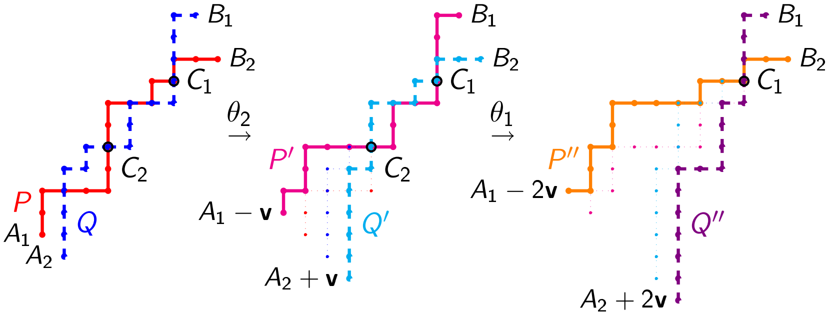

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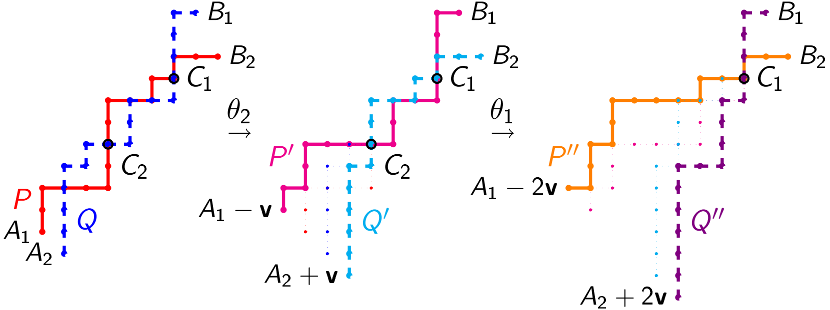

In this example, we have a bijection

$$\theta_1 \circ \theta_2 : \mathcal{P}^{\geq 2}_{A_1 \to B_2, A_2 \to B_1} \to \mathcal{P}^{\geq 0}_{A_1 - 2\mathbf{v} \to B_2, A_2 + 2\mathbf{v} \to B_1}.$$

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The pairs of paths in the image are easy to enumerate.

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The pairs of paths in the image are easy to enumerate. In this case, we obtain

$$H^{\geq 2}_{A_1 \to B_2, A_2 \to B_1}(q) = q^{2(2+x_2-x_1)} \begin{bmatrix} u_2 - x_1 + v_2 - y_1 \\ u_2 - x_1 + 2 \end{bmatrix}_q \begin{bmatrix} u_1 - x_2 + v_1 - y_2 \\ u_1 - x_2 - 2 \end{bmatrix}_q,$$

where $A_1 \prec A_2$ and $B_1 \prec B_2$.

Bijections for paths crossing a horizontal line

For the problem of a path crossing a horizontal line, we define similar bijections τ and σ. They apply to paths with U and D steps ending on the x-axis, and they fix the right endpoint.

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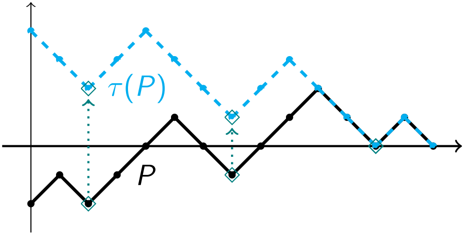
τ reflects the valleys along the x-axis:



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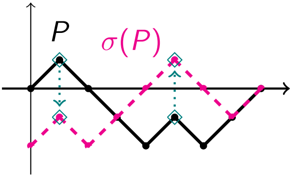

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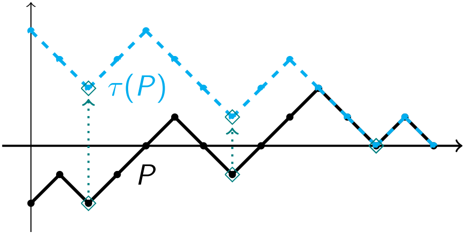

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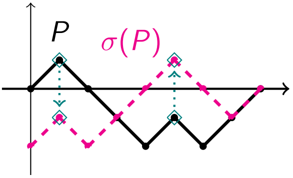

$\mathrm{maj}(\tau(P)) = \mathrm{maj}(P)$, $\mathrm{maj}(\sigma(P)) = \mathrm{maj}(P) + \#U - \#D - 1$

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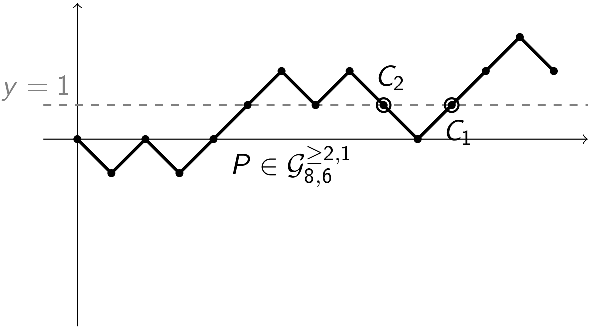

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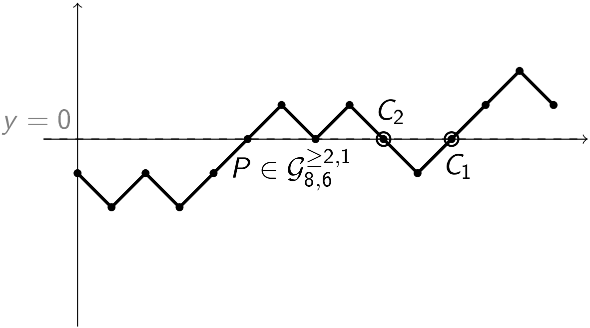

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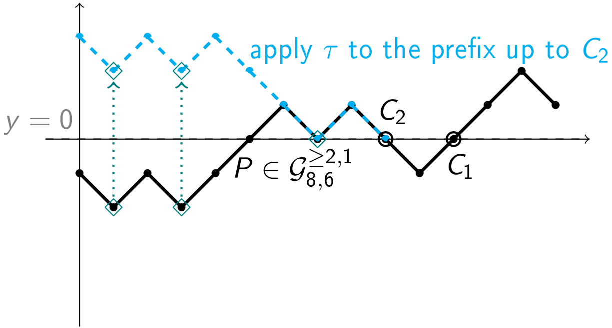

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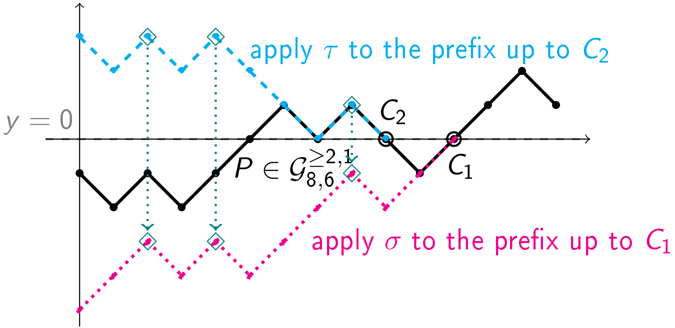

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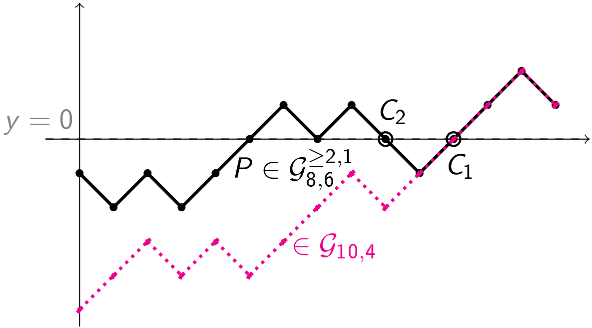

In this case, we get a bijection $\mathcal{G}_{a,b}^{\geq 2,\ell} \to \mathcal{G}_{a+2,b-2}$ that decreases maj by $\ell + 3$. The paths in the image are easy to count.

Further refinements

Our results can be refined by keeping track of the number of descents (i.e., DU or EN corners).

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Our results can be refined by keeping track of the number of descents (i.e., *DU* or *EN* corners). Here are some sample results:

If $0 < \ell < a - b$, then

$$\sum_{P \in \mathcal{G}_{a,b}^{\geq 2m,\ell}} t^{\mathrm{des}(P)} q^{\mathrm{maj}(P)} = \sum_k t^k q^{k^2+m(m+1+\ell)} \begin{bmatrix} a \\ k-m \end{bmatrix}_q \begin{bmatrix} b \\ k+m \end{bmatrix}_q .$$

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If $A_1 \prec A_2$, $B_1 \prec B_2$, and $x_1 + y_1 = x_2 + y_2$, then, for all $m \geq 0$,

$$\sum_{(P,Q) \in \mathcal{P}^{\geq r}_{A_1 \to B_2, A_2 \to B_1}} t^{\mathrm{des}(P)+\mathrm{des}(Q)} q^{\mathrm{maj}(P)+\mathrm{maj}(Q)}$$

$$= q^{2m(2m+x_2-x_1)} \cdot \left(\sum_k t^k q^{k(k+2m)} \begin{bmatrix} u_2 - x_1 \\ k \end{bmatrix}_q \begin{bmatrix} v_2 - y_1 \\ k+2m \end{bmatrix}_q \right)$$

$$\cdot \left(\sum_k t^k q^{k(k-2m)} \begin{bmatrix} u_1 - x_2 \\ k \end{bmatrix}_q \begin{bmatrix} v_1 - y_2 \\ k-2m \end{bmatrix}_q \right).$$

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Our bijections $\bar{\tau}$, $\bar{\sigma}$, σ do not behave well with respect to the number of descents.

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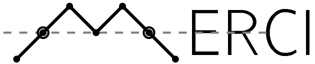
Instead, we prove these refinements using different bijections that rely on Krattenthaler's two-rowed arrays.



arXiv:2106.09878