The probability of avoiding consecutive patterns in the Mallows distribution

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Joint work with Harry Crane and Stephen DeSalvo

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Consecutive patterns

$$\pi = \pi_1\pi_2\dots\pi_n \in \mathcal{S}_n, \quad \sigma = \sigma_1\sigma_2\dots\sigma_m \in \mathcal{S}_m.$$

Definition

π contains σ as a consecutive pattern if π has a subsequence of adjacent entries $\pi_i\pi_{i+1}\dots\pi_{i+m-1}$ in the same relative order as $\sigma_1\dots\sigma_m$; otherwise π avoids σ.

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42531 contains 132, but 25134 avoids 132.

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Disregarding these implicit appearances, the systematic study of consecutive patterns in permutations started about 20 years ago.

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Formulas for $F_\sigma(z)$ are known for some patterns.

Example

$$F_{132}(z) = \left(1 - \int_0^z e^{-t^2/2}\, dt\right)^{-1}.$$

$$F_{1234}(z) = \frac{2}{\cos z - \sin z + e^{-z}}.$$

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It is convenient to define $\omega_\sigma(z) = F_\sigma(z)^{-1}$.

Exact enumeration

Theorem (E.–Noy '01)

For $\sigma = 12\ldots m$, $\omega = \omega_\sigma(z)$ *satisfies*

$$\omega^{(m-1)} + \omega^{(m-2)} + \cdots + \omega' + \omega = 0.$$

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Let $\sigma \in \mathcal{S}_m$ *be non-overlapping with* $\sigma_1 = 1$, $\sigma_m = b$. *Then* $\omega = \omega_\sigma(z)$ *satisfies*

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The analogous question in the case of "classical" (i.e. non-consecutive) patterns is still open.

Garrabrant–Pak '15 prove that some generating functions for permutations avoiding **sets** of classical patterns are not D-finite.

Asymptotic behavior

Theorem (E. '05)

For every σ, the limit

$$\rho_\sigma := \lim_{n\to\infty} \left(\frac{|\mathcal{S}_n(\sigma)|}{n!} \right)^{1/n} \qquad \textit{exists.}$$

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Theorem (Ehrenborg–Kitaev–Perry '11)

For every σ,

$$\frac{|\mathcal{S}_n(\sigma)|}{n!} = \gamma_\sigma \rho_\sigma^n + O(\delta^n),$$

for some constants γ_σ and $\delta < \rho_\sigma$.

The proof uses methods from spectral theory.

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No known analogues for classical (i.e. non-consecutive) patterns.

Inversions

Definition

An inversion of $\pi \in \mathcal{S}_n$ is a pair (i, j) with $i < j$ and $\pi_i > \pi_j$.
Let $\mathrm{inv}(\pi)$ = number of inversions of π.

Example: $\mathrm{inv}(3142) = 3$, since $3 > 1$, $3 > 2$ and $4 > 2$.

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Definition (Mallows '57)

Fix a real parameter $q>0$. The Mallows distribution on $\mathcal{S}_n$ assigns probability

$$\frac{q^{\text{inv}(\pi)}}{[n]_q!}$$

to each $\pi \in \mathcal{S}_n$, where
$[n]_q! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$.

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It is a canonical statistical model for ranking data.

Generating functions again

The probability that a random permutation from the Mallows distribution avoids σ is

$$P_n(\sigma, q) := \sum_{\pi \in \mathcal{S}_n(\sigma)} \frac{q^{\mathrm{inv}(\pi)}}{[n]_q!}.$$

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$$F_\sigma(q, z) := \sum_{n \geq 0} P_n(\sigma, q) z^n = \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n(\sigma)} q^{\mathrm{inv}(\pi)} \frac{z^n}{[n]_q!}.$$

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Denoting by σ^r and σ^c the reversal and the complement of σ, we have

$$F_\sigma(q, z) = F_{\sigma^r}(1/q, z) = F_{\sigma^c}(1/q, z) = F_{\sigma^{rc}}(q, z).$$

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We refine the cluster method to keep track of inversions, so it applies to the Mallows distribution.

The generalized cluster method

Let $c_\sigma(i,k,n) = \#\{k\text{-clusters } \pi \in \mathcal{S}_n \text{ w.r.t. } \sigma \text{ with } \mathrm{inv}(\pi) = i\}$.

Define the cluster generating function

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The proof combines inclusion-exclusion with some properties of inv.

Growth rates exist

Theorem (Crane–DeSalvo–E. '18)

For every $q > 0$ and every pattern σ, the limit

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To plot $\rho_\sigma(q)$ as a function of q for $q \in (0, \infty)$, we use the change of variables $x = \frac{q-1}{q+1}$, so that $x \in (-1, 1)$.

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$\rho_\sigma(q)^{-1}$ is the radius of convergence of $F_\sigma(q, z)$ as a function of a complex variable z.

Monotone patterns

Theorem (E. '16)

$$F_{12\ldots m}(q,z) = \left(\sum_{j\geq 0} \frac{z^{jm}}{[jm]_q!} - \sum_{j\geq 0} \frac{z^{jm+1}}{[jm+1]_q!} \right)^{-1}$$

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We can use this to approximate $\rho_{12\dots m}(q)$, which is the reciprocal of the smallest positive zero of the denominator.

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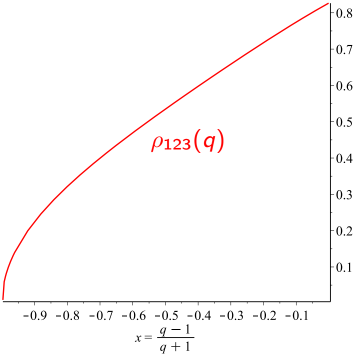

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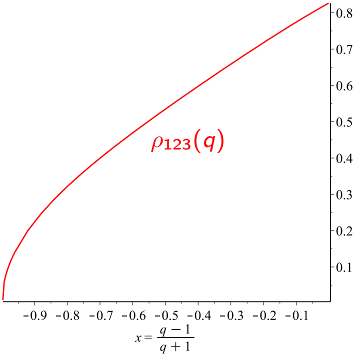

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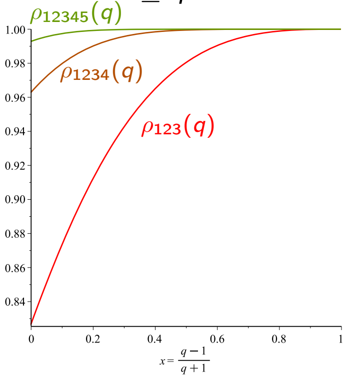

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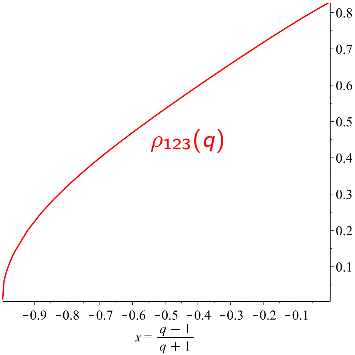

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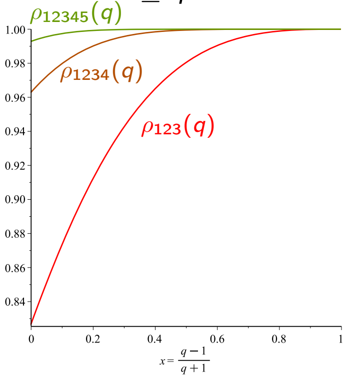

Conjecture (Crane–DeSalvo–E. '18)

If $q < q'$, *then* $P_n(12\ldots m, q) < P_n(12\ldots m, q')$
and $\rho_{12\ldots m}(q) < \rho_{12\ldots m}(q')$.

Non-overlapping patterns with $\sigma_1 = 1$

Theorem (Rawlings '07, E. 16)

Let $\sigma = \sigma_1 \dots \sigma_m$ *be non-overlapping with* $\sigma_1 = 1$, $\sigma_m = b$. *Then*

$$F_\sigma(q,z) = \left(1 - z - \sum_{k\geq 1} \prod_{j=1}^{k-1} \binom{j(m-1)+m-b}{m-b}_q \frac{q^{k\,\mathrm{inv}(\sigma)}(-1)^k z^{k(m-1)+1}}{[k(m-1)+1]_q!}\right)^{-1}.$$

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Again, after some painful calculations, we can approximate the smallest positive zero of the denominator to get $\rho_\sigma(q)^{-1}$.

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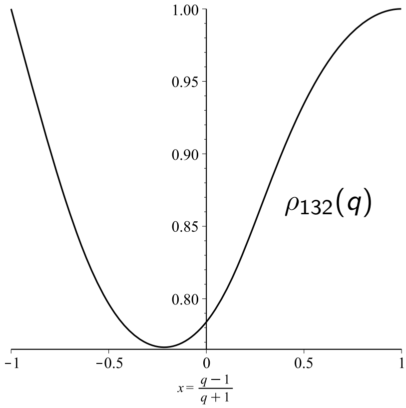

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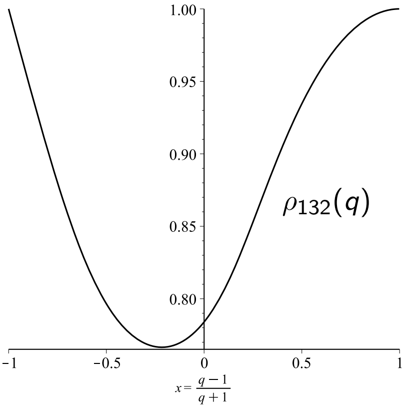

The minimum of $\rho(132, q)$ is attained at $q_0 \approx 0.6447045$, giving a growth rate of $\rho(132, q_0) \approx 0.7665452$.

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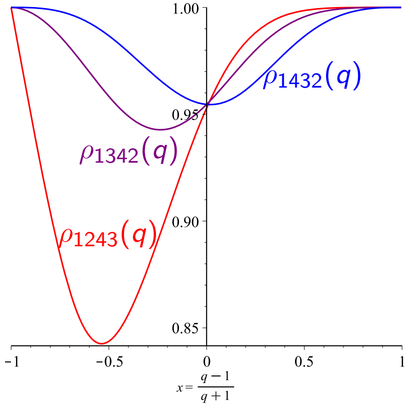

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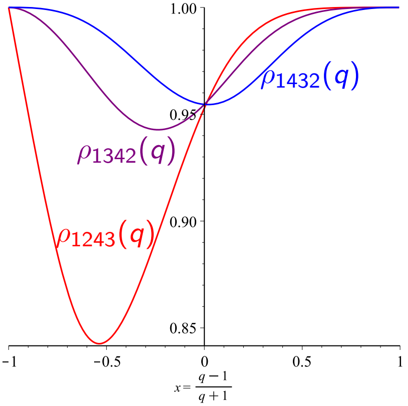

For $q = 1$, we have $\rho_{1243}(1) < \rho_{1342}(1) = \rho_{1432}(1)$.

Comparisons among patterns

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Conjecture (Crane–DeSalvo–E. '18)

For $q \geq 1$ and every n,

$$P_n(231, q) \leq P_n(132, q).$$

Bounds on $\rho_\sigma(q)$

Even for patterns for which $F_\sigma(q, z)$ is unknown, we have general upper and lower bounds on the growth rate $\rho_\sigma(q)$.

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Recall: $c_\sigma(i,2,n) = \#\{\text{2-clusters } \pi \in \mathcal{S}_n \text{ w.r.t. } \sigma \text{ with } \mathrm{inv}(\pi) = i\}$.

$$T_\sigma(q) := \sum_{i,n} c_\sigma(i,2,n) \frac{q^i}{[n]_q!}.$$

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Proposition (Crane–DeSalvo–E. '18)

Fix $m \geq 3$, $\sigma \in S_m$ and $q > 0$. Then

$$\rho_\sigma(q) \leq \exp\left(-\frac{q^{\text{inv}(\sigma)}}{[m]_q!} + \exp\left(4(m-1)\frac{q^{\text{inv}(\sigma)}}{[m]_q!} \right) T_\sigma(q) \right).$$

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Proposition (Crane–DeSalvo–E. '18)

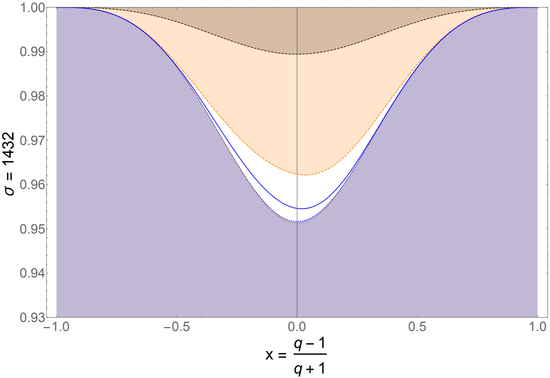
Fix $m \geq 3$, $\sigma \in S_m$, $q > 0$. *Then*

$$\rho_\sigma(q) \geq 1 - \frac{q^{\mathrm{inv}(\sigma)}}{[m]_q!} \exp\left(\frac{1}{2}\left(1 - \frac{q^{\mathrm{inv}(\sigma)}}{[m]_q!} - \sqrt{1 - (4m-2)\frac{q^{\mathrm{inv}(\sigma)}}{[m]_q!} + \frac{q^{2\,\mathrm{inv}(\sigma)}}{[m]_q!^2}}\right)\right).$$

Bounds on $\rho_\sigma(q)$

Example: $\sigma = 1432$.

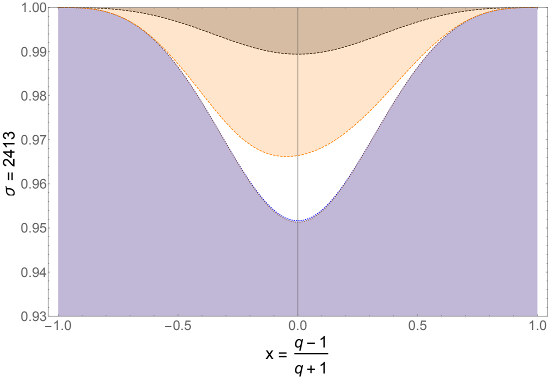
The blue curve is the actual $\rho_{1432}(q)$ computed earlier.



Bounds on $\rho_\sigma(q)$

Example: $\sigma = 2413$.

Neither $F_{2413}(q,z)$ nor the growth rate $\rho_{2413}(q)$ are known.



Thank you