Descents on noncrossing and nonnesting permutations

Sergi Elizalde

Dartmouth College

Queen Mary, University of London
February 2023

Outline

1. Background: descents, Eulerian polynomials, Stirling permutations.
2. Noncrossing (or quasi-Stirling) permutations.
3. Nonnesting permutations.

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\dots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\dots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,
- ascent if $\pi_i < \pi_{i+1}$ or $i = 0$,

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\ldots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,
- ascent if $\pi_i < \pi_{i+1}$ or $i = 0$,
- plateau if $\pi_i = \pi_{i+1}$.

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\dots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,
- ascent if $\pi_i < \pi_{i+1}$ or $i = 0$,
- plateau if $\pi_i = \pi_{i+1}$.

$\mathrm{des}(\pi)$ = number of descents of π
$\mathrm{asc}(\pi)$ = number of ascents of π
$\mathrm{plat}(\pi)$ = number of plateaus of π

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\dots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,
- ascent if $\pi_i < \pi_{i+1}$ or $i = 0$,
- plateau if $\pi_i = \pi_{i+1}$.

$\mathrm{des}(\pi)$ = number of descents of π
$\mathrm{asc}(\pi)$ = number of ascents of π
$\mathrm{plat}(\pi)$ = number of plateaus of π

Example

$\mathrm{des}(36522131) =$
$\mathrm{asc}(36522131) =$
$\mathrm{plat}(36522131) =$

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\ldots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,
- ascent if $\pi_i < \pi_{i+1}$ or $i = 0$,
- plateau if $\pi_i = \pi_{i+1}$.

$\mathrm{des}(\pi)$ = number of descents of π
$\mathrm{asc}(\pi)$ = number of ascents of π
$\mathrm{plat}(\pi)$ = number of plateaus of π

Example

$\mathrm{des}(36{\cdot}5{\cdot}22{\cdot}13{\cdot}1{\cdot}) =$
$\mathrm{asc}(36522131) =$
$\mathrm{plat}(36522131) =$

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\ldots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,
- ascent if $\pi_i < \pi_{i+1}$ or $i = 0$,
- plateau if $\pi_i = \pi_{i+1}$.

$\mathrm{des}(\pi)$ = number of descents of π
$\mathrm{asc}(\pi)$ = number of ascents of π
$\mathrm{plat}(\pi)$ = number of plateaus of π

Example

$\mathrm{des}(36{\cdot}5{\cdot}22{\cdot}13{\cdot}1{\cdot}) = 5$
$\mathrm{asc}(36522131) =$
$\mathrm{plat}(36522131) =$

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\dots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,
- ascent if $\pi_i < \pi_{i+1}$ or $i = 0$,
- plateau if $\pi_i = \pi_{i+1}$.

$\mathrm{des}(\pi)$ = number of descents of π
$\mathrm{asc}(\pi)$ = number of ascents of π
$\mathrm{plat}(\pi)$ = number of plateaus of π

Example

$\mathrm{des}(36{\cdot}5{\cdot}22{\cdot}13{\cdot}1{\cdot}) = 5$
$\mathrm{asc}({\cdot}3{\cdot}65221{\cdot}31) = 3$
$\mathrm{plat}(36522131) =$

Descents and plateaus

Definition

Let $\pi = \pi_1\pi_2\ldots\pi_m$ be a sequence of positive integers. We say that i is a

- descent if $\pi_i > \pi_{i+1}$ or $i = m$,
- ascent if $\pi_i < \pi_{i+1}$ or $i = 0$,
- plateau if $\pi_i = \pi_{i+1}$.

$\mathrm{des}(\pi)$ = number of descents of π
$\mathrm{asc}(\pi)$ = number of ascents of π
$\mathrm{plat}(\pi)$ = number of plateaus of π

Example

$\mathrm{des}(36{\cdot}5{\cdot}22{\cdot}13{\cdot}1{\cdot}) = 5$
$\mathrm{asc}({\cdot}3{\cdot}65221{\cdot}31) = 3$
$\mathrm{plat}(3652{\cdot}2131) = 1$

Eulerian polynomials

$[n] = \{1, 2, \ldots, n\}$
$\mathcal{S}_n$ = set of permutations of $[n]$

Eulerian polynomials

$[n] = \{1, 2, \ldots, n\}$
$\mathcal{S}_n =$ set of permutations of $[n]$

Definition

Eulerian polynomials:

$$A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{des}(\pi)}$$

Eulerian polynomials

$[n] = \{1, 2, \ldots, n\}$
$\mathcal{S}_n$ = set of permutations of $[n]$

Definition

Eulerian polynomials:

$$A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{des}(\pi)}$$

Example

$A_1(t) = t$ — $1\cdot$

$A_2(t) = t + t^2$ — $12\cdot$, $2\cdot1\cdot$

$A_3(t) = t + 4t^2 + t^3$ — $123\cdot$, $13\cdot2\cdot$, $2\cdot13\cdot$, $23\cdot1\cdot$, $3\cdot12\cdot$, $3\cdot2\cdot1\cdot$

$A_4(t) = t + 11t^2 + 11t^3 + t^4$ — $\ldots$

Eulerian polynomials

$[n] = \{1, 2, \ldots, n\}$
$\mathcal{S}_n$ = set of permutations of $[n]$

Definition

Eulerian polynomials:

$$A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{des}(\pi)}$$

Example

$A_1(t) = t$ 1·

$A_2(t) = t + t^2$ 12·, 2·1·

$A_3(t) = t + 4t^2 + t^3$ 123·, 13·2·, 2·13·, 23·1·, 3·12·, 3·2·1·

$A_4(t) = t + 11t^2 + 11t^3 + t^4$ ...

These polynomials appear in work of Euler from 1755.

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2\,(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1.2.3\,(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1.2.3.4\,(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1.2.3.4.5\,(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1.2.3.4.5.6\,(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1.2.3.4.5.6.7\,(p-1)^7}$$

&c.

L. Euler, 1755.

Eulerian Polynomials

$$\frac{A_n(p)/p}{n!(p-1)^n} \quad (1 \le n \le 7)$$

Eulerian polynomials

Euler was considering the series

$$\sum_{m\geq 0} m t^m = \frac{t}{(1-t)^2}$$

$$\sum_{m\geq 0} m^2 t^m = \frac{t+t^2}{(1-t)^3}$$

$$\sum_{m\geq 0} m^3 t^m = \frac{t+4t^2+t^3}{(1-t)^4}$$

$$\sum_{m\geq 0} m^4 t^m = \frac{t+11t^2+11t^3+t^4}{(1-t)^5} \qquad \cdots$$

Eulerian polynomials

Euler was considering the series

$$\sum_{m\geq 0} m t^m = \frac{t}{(1-t)^2}$$

$$\sum_{m\geq 0} m^2 t^m = \frac{t+t^2}{(1-t)^3}$$

$$\sum_{m\geq 0} m^3 t^m = \frac{t+4t^2+t^3}{(1-t)^4}$$

$$\sum_{m\geq 0} m^4 t^m = \frac{t+11t^2+11t^3+t^4}{(1-t)^5} \qquad \dots$$

In general,

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Eulerian polynomials

Euler was considering the series

$$\sum_{m\geq 0} mt^m = \frac{t}{(1-t)^2}$$

$$\sum_{m\geq 0} m^2 t^m = \frac{t+t^2}{(1-t)^3}$$

$$\sum_{m\geq 0} m^3 t^m = \frac{t+4t^2+t^3}{(1-t)^4}$$

$$\sum_{m\geq 0} m^4 t^m = \frac{t+11t^2+11t^3+t^4}{(1-t)^5} \qquad \dots$$

In general,

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}.$$

This can be proved by induction on n, differentiating both sides.

Generating function for Eulerian polynomials

Let

$$A(t,z)=\sum_{n\geq 0} A_n(t)\frac{z^n}{n!}$$

be the exponential generating function (EGF) for the Eulerian polynomials.

Generating function for Eulerian polynomials

Let

$$A(t,z)=\sum_{n\geq 0}A_n(t)\frac{z^n}{n!}$$

be the exponential generating function (EGF) for the Eulerian polynomials.

It is known that

$$A(t,z)=\frac{1-t}{1-te^{(1-t)z}}.$$

Stirling numbers

Definition

The Stirling number of the second kind $S(n,k)$ is the number of partitions of the set $\{1,2,\ldots,n\}$ into k blocks.

Stirling numbers

Definition

The Stirling number of the second kind $S(n, k)$ is the number of partitions of the set $\{1, 2, \ldots, n\}$ into k blocks.

Example: $S(3, 2) = 3$. $\{1, 2\} \cup \{3\}$, $\{1, 3\} \cup \{2\}$, $\{2, 3\} \cup \{1\}$.

Stirling numbers

Definition

The Stirling number of the second kind $S(n,k)$ is the number of partitions of the set $\{1,2,\ldots,n\}$ into k blocks.

Example: $S(3,2)=3$. $\{1,2\}\cup\{3\}$, $\{1,3\}\cup\{2\}$, $\{2,3\}\cup\{1\}$.

In 1978, Gessel and Stanley considered the series

$$\sum_{m\geq 0} S(m+1,m)\,t^m = \frac{t}{(1-t)^3}$$

$$\sum_{m\geq 0} S(m+2,m)\,t^m = \frac{t+2t^2}{(1-t)^5}$$

$$\sum_{m\geq 0} S(m+3,m)\,t^m = \frac{t+8t^2+6t^3}{(1-t)^7}$$

$$\sum_{m\geq 0} S(m+4,m)\,t^m = \frac{t+22t^2+58t^3+24t^4}{(1-t)^9} \qquad \ldots$$

Stirling numbers

Definition

The Stirling number of the second kind $S(n,k)$ is the number of partitions of the set $\{1,2,\ldots,n\}$ into k blocks.

Example: $S(3,2)=3$. $\{1,2\}\cup\{3\}$, $\{1,3\}\cup\{2\}$, $\{2,3\}\cup\{1\}$.

In 1978, Gessel and Stanley considered the series

$$\sum_{m\geq 0} S(m+1,m)\,t^m = \frac{t}{(1-t)^3}$$

$$\sum_{m\geq 0} S(m+2,m)\,t^m = \frac{t+2t^2}{(1-t)^5}$$

$$\sum_{m\geq 0} S(m+3,m)\,t^m = \frac{t+8t^2+6t^3}{(1-t)^7}$$

$$\sum_{m\geq 0} S(m+4,m)\,t^m = \frac{t+22t^2+58t^3+24t^4}{(1-t)^9} \qquad \ldots$$

What are the polynomials in the numerator? Why positive coefficients?

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1, 1, 2, 2, \ldots, n, n\}$.

Definition (Gessel–Stanley '78)

A Stirling permutation is a permutation of $[n] \sqcup [n]$ that avoids the pattern 212.

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1, 1, 2, 2, \ldots, n, n\}$.

Definition (Gessel–Stanley '78)

A Stirling permutation is a permutation of $[n] \sqcup [n]$ that avoids the pattern 212.

In other words, if $\pi_1 \pi_2 \ldots \pi_{2n}$ is a Stirling permutation, there do not exist $i < j < k$ such that $\pi_i = \pi_k > \pi_j$.

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1, 1, 2, 2, \ldots, n, n\}$.

Definition (Gessel–Stanley '78)

A Stirling permutation is a permutation of $[n] \sqcup [n]$ that avoids the pattern 212.

In other words, if $\pi_1\pi_2 \ldots \pi_{2n}$ is a Stirling permutation, there do not exist $i < j < k$ such that $\pi_i = \pi_k > \pi_j$.

$\mathcal{Q}_n$ = set of Stirling permutations of $[n] \sqcup [n]$.

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1, 1, 2, 2, \ldots, n, n\}$.

Definition (Gessel–Stanley '78)

A Stirling permutation is a permutation of $[n] \sqcup [n]$ that avoids the pattern 212.

In other words, if $\pi_1\pi_2\ldots\pi_{2n}$ is a Stirling permutation, there do not exist $i < j < k$ such that $\pi_i = \pi_k > \pi_j$.

$\mathcal{Q}_n$ = set of Stirling permutations of $[n] \sqcup [n]$.

Example

$13324421 \in \mathcal{Q}_4$,

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1, 1, 2, 2, \ldots, n, n\}$.

Definition (Gessel–Stanley '78)

A Stirling permutation is a permutation of $[n] \sqcup [n]$ that avoids the pattern 212.

In other words, if $\pi_1\pi_2\ldots\pi_{2n}$ is a Stirling permutation, there do not exist $i < j < k$ such that $\pi_i = \pi_k > \pi_j$.

$\mathcal{Q}_n$ = set of Stirling permutations of $[n] \sqcup [n]$.

Example

$13324421 \in \mathcal{Q}_4$, $312321 \notin \mathcal{Q}_3$,

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1, 1, 2, 2, \dots, n, n\}$.

Definition (Gessel–Stanley '78)

A Stirling permutation is a permutation of $[n] \sqcup [n]$ that avoids the pattern 212.

In other words, if $\pi_1\pi_2\dots\pi_{2n}$ is a Stirling permutation, there do not exist $i < j < k$ such that $\pi_i = \pi_k > \pi_j$.

$\mathcal{Q}_n$ = set of Stirling permutations of $[n] \sqcup [n]$.

Example

$13324421 \in \mathcal{Q}_4$, $\quad 312321 \notin \mathcal{Q}_3$, $\quad \mathcal{Q}_2 = \{1122, 1221, 2211\}$.

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1, 1, 2, 2, \ldots, n, n\}$.

Definition (Gessel–Stanley '78)

A Stirling permutation is a permutation of $[n] \sqcup [n]$ that avoids the pattern 212.

In other words, if $\pi_1 \pi_2 \ldots \pi_{2n}$ is a Stirling permutation, there do not exist $i < j < k$ such that $\pi_i = \pi_k > \pi_j$.

$\mathcal{Q}_n$ = set of Stirling permutations of $[n] \sqcup [n]$.

Example

$13324421 \in \mathcal{Q}_4$, $312321 \notin \mathcal{Q}_3$, $\mathcal{Q}_2 = \{1122, 1221, 2211\}$.

We have $|\mathcal{Q}_n| = (2n-1) \cdot (2n-3) \cdot \cdots \cdot 3 \cdot 1$, since every permutation in $\mathcal{Q}_n$ can be obtained by inserting nn into one of the $2n-1$ spaces of a permutation in $\mathcal{Q}_{n-1}$.

Stirling polynomials

Definition (Gessel–Stanley '78)

Stirling polynomials: $$Q_n(t) = \sum_{\pi \in \mathcal{Q}_n} t^{\operatorname{des}(\pi)}$$

Stirling polynomials

Definition (Gessel–Stanley '78)

Stirling polynomials:
$$Q_n(t) = \sum_{\pi \in \mathcal{Q}_n} t^{\mathrm{des}(\pi)}$$

Example

$$Q_1(t) = t \qquad 11\cdot$$
$$Q_2(t) = t + 2t^2 \qquad 1122\cdot,\ 122\cdot1\cdot,\ 22\cdot11\cdot$$
$$Q_3(t) = t + 8t^2 + 6t^3 \qquad \ldots$$

Stirling polynomials

Definition (Gessel–Stanley '78)

Stirling polynomials: $$Q_n(t) = \sum_{\pi \in \mathcal{Q}_n} t^{\mathrm{des}(\pi)}$$

Example

$$Q_1(t) = t \qquad 11\cdot$$

$$Q_2(t) = t + 2t^2 \qquad 1122\cdot,\ 122\cdot1\cdot,\ 22\cdot11\cdot$$

$$Q_3(t) = t + 8t^2 + 6t^3 \qquad \ldots$$

Theorem (Gessel–Stanley '78)

$$\sum_{m \geq 0} S(m+n, m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}}.$$

Literature on Stirling permutations

There is an extensive literature on Stirling permutations. Some work relevant to this talk:

- Bóna '08: $Q_n(t)$ also gives the enumeration of $\mathcal{Q}_n$ by the number of plateaus, that is, positions i such that $\pi_i = \pi_{i+1}$.

Literature on Stirling permutations

There is an extensive literature on Stirling permutations. Some work relevant to this talk:

- Bóna '08: $Q_n(t)$ also gives the enumeration of $\mathcal{Q}_n$ by the number of plateaus, that is, positions i such that $\pi_i = \pi_{i+1}$.
- Brenti '89, Bóna '08: $Q_n(t)$ has only real roots, and the distribution of des on $\mathcal{Q}_n$ is asymptotically normal.

Literature on Stirling permutations

There is an extensive literature on Stirling permutations. Some work relevant to this talk:

- Bóna '08: $Q_n(t)$ also gives the enumeration of $\mathcal{Q}_n$ by the number of plateaus, that is, positions i such that $\pi_i = \pi_{i+1}$.
- Brenti '89, Bóna '08: $Q_n(t)$ has only real roots, and the distribution of des on $\mathcal{Q}_n$ is asymptotically normal.
- Janson '08: The joint distribution of ascents, descents and plateaus on $\mathcal{Q}_n$ is asymptotically normal.

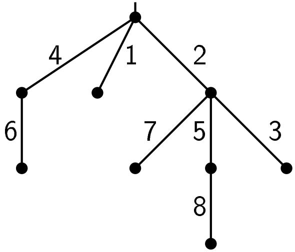
Literature on Stirling permutations

There is an extensive literature on Stirling permutations. Some work relevant to this talk:

- Bóna '08: $Q_n(t)$ also gives the enumeration of $\mathcal{Q}_n$ by the number of plateaus, that is, positions i such that $\pi_i = \pi_{i+1}$.
- Brenti '89, Bóna '08: $Q_n(t)$ has only real roots, and the distribution of des on $\mathcal{Q}_n$ is asymptotically normal.
- Janson '08: The joint distribution of ascents, descents and plateaus on $\mathcal{Q}_n$ is asymptotically normal.
- The coefficients of $Q_n(t)$ are sometimes called second-order Eulerian numbers.

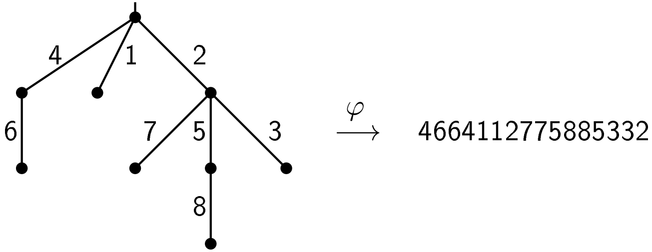
Stirling permutations and trees

$\mathcal{I}_n$ = set of increasing edge-labeled plane rooted trees with n edges.



Stirling permutations and trees

$\mathcal{I}_n$ = set of increasing edge-labeled plane rooted trees with n edges.

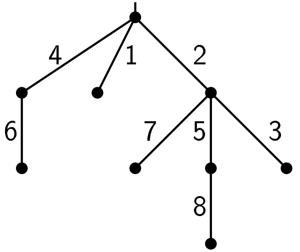


Theorem (Koganov '96, Janson '08)

There is a bijection $\varphi : \mathcal{I}_n \longrightarrow \mathcal{Q}_n$ obtained by traversing the edges of the tree along a depth-first walk from left to right, and recording their labels.

Stirling permutations and trees

$\mathcal{I}_n$ = set of increasing edge-labeled plane rooted trees with n edges.



$\xrightarrow{\varphi}$ 4664112775885332

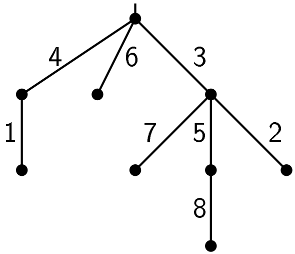
Theorem (Koganov '96, Janson '08)

There is a bijection $\varphi : \mathcal{I}_n \longrightarrow \mathcal{Q}_n$ obtained by traversing the edges of the tree along a depth-first walk from left to right, and recording their labels.

If we remove the increasing condition on the trees, what is the image of φ?

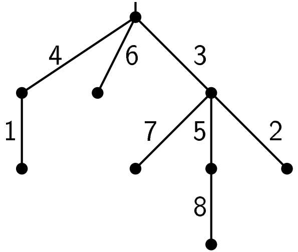
Removing the increasing condition

$\mathcal{T}_n$ = set of edge-labeled plane rooted trees with n edges.



Removing the increasing condition

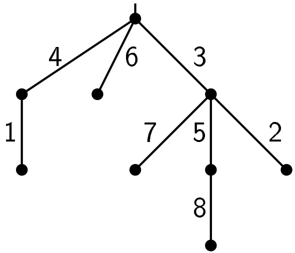
$\mathcal{T}_n$ = set of edge-labeled plane rooted trees with n edges.



$\xrightarrow{\varphi}$ 4114663775885223

Removing the increasing condition

$\mathcal{T}_n$ = set of edge-labeled plane rooted trees with n edges.



$\xrightarrow{\varphi}$ 4114663775885223

Theorem (Archer–Gregory–Pennington–Slayden '19)

φ is a bijection between $\mathcal{T}_n$ and $\overline{\mathcal{Q}}_n$ (to be defined on the next page).

Noncrossing permutations

Definition (Archer–Gregory–Pennington–Slayden '19)

A quasi-Stirling (or noncrossing) permutation is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121.

Definition (Archer–Gregory–Pennington–Slayden '19)

A quasi-Stirling (or noncrossing) permutation is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$.

Noncrossing permutations

Definition (Archer–Gregory–Pennington–Slayden '19)

A quasi-Stirling (or noncrossing) permutation is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$.

$\overline{\mathcal{Q}}_n$ = set of noncrossing permutations of $[n] \sqcup [n]$.

Noncrossing permutations

Definition (Archer–Gregory–Pennington–Slayden '19)

A quasi-Stirling (or noncrossing) permutation is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$.

$\overline{\mathcal{Q}}_n$ = set of noncrossing permutations of $[n] \sqcup [n]$.

Example

$4431152253 \in \overline{\mathcal{Q}}_5$,

Noncrossing permutations

Definition (Archer–Gregory–Pennington–Slayden '19)

A quasi-Stirling (or noncrossing) permutation is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$.

$\overline{\mathcal{Q}}_n$ = set of noncrossing permutations of $[n] \sqcup [n]$.

Example

$4431152253 \in \overline{\mathcal{Q}}_5$, $312321 \notin \overline{\mathcal{Q}}_3$,

Noncrossing permutations

Definition (Archer–Gregory–Pennington–Slayden '19)

A quasi-Stirling (or noncrossing) permutation is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$.

$\overline{\mathcal{Q}}_n$ = set of noncrossing permutations of $[n] \sqcup [n]$.

Example

$4431152253 \in \overline{\mathcal{Q}}_5$, $\quad 312321 \notin \overline{\mathcal{Q}}_3$, $\quad \overline{\mathcal{Q}}_2 = \{1122, 1221, 2211, 2112\}$.

Noncrossing permutations

Definition (Archer–Gregory–Pennington–Slayden '19)

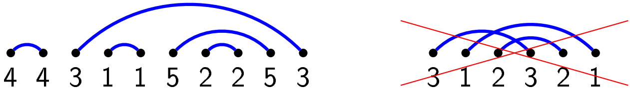
A quasi-Stirling (or noncrossing) permutation is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$.

$\overline{\mathcal{Q}}_n$ = set of noncrossing permutations of $[n] \sqcup [n]$.

Example

$4431152253 \in \overline{\mathcal{Q}}_5, \qquad 312321 \notin \overline{\mathcal{Q}}_3, \qquad \overline{\mathcal{Q}}_2 = \{1122, 1221, 2211, 2112\}.$



They are in bijection with labeled noncrossing matchings.

Noncrossing permutations

Definition (Archer–Gregory–Pennington–Slayden '19)

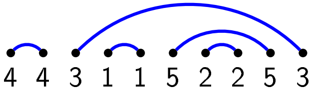
A quasi-Stirling (or noncrossing) permutation is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121.

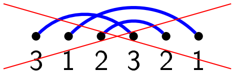
Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$.

$\overline{\mathcal{Q}}_n$ = set of noncrossing permutations of $[n] \sqcup [n]$.

Example

$4431152253 \in \overline{\mathcal{Q}}_5, \qquad 312321 \notin \overline{\mathcal{Q}}_3, \qquad \overline{\mathcal{Q}}_2 = \{1122, 1221, 2211, 2112\}.$





They are in bijection with labeled noncrossing matchings. It follows that

$$|\overline{\mathcal{Q}}_n| = n!\mathrm{Cat}_n = \frac{(2n)!}{(n+1)!}.$$

Noncrossing permutations with most descents

One can show that, for any $\pi \in \overline{\mathcal{Q}}_n$, we have $\mathrm{des}(\pi) \leq n$.

Noncrossing permutations with most descents

One can show that, for any $\pi \in \overline{\mathcal{Q}}_n$, we have $\operatorname{des}(\pi) \leq n$.

Theorem (E. '21)

The number of $\pi \in \overline{\mathcal{Q}}_n$ with $\operatorname{des}(\pi) = n$ is equal to $(n+1)^{n-1}$.

This had been conjectured by Archer–Gregory–Pennington–Slayden '19.

One can show that, for any $\pi \in \overline{\mathcal{Q}}_n$, we have $\mathrm{des}(\pi) \leq n$.

Theorem (E. '21)

The number of $\pi \in \overline{\mathcal{Q}}_n$ with $\mathrm{des}(\pi) = n$ is equal to $(n+1)^{n-1}$.

This had been conjectured by Archer–Gregory–Pennington–Slayden '19.

Prof idea:

- Translate the statistic des into a statistic on trees via the bijection φ.
- Show that trees that maximize this statistic are in bijection with Cayley trees, which are counted by $(n+1)^{n-1}$.

Descents on noncrossing permutations

We are interested not only in how many permutations maximize des, but more generally in the distribution of des over $\overline{\mathcal{Q}}_n$.

Descents on noncrossing permutations

We are interested not only in how many permutations maximize des, but more generally in the distribution of des over $\overline{\mathcal{Q}}_n$.

Define the quasi-Stirling polynomials

$$\overline{Q}_n(t) = \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\mathrm{des}(\pi)}.$$

Descents on noncrossing permutations

We are interested not only in how many permutations maximize des, but more generally in the distribution of des over $\overline{\mathcal{Q}}_n$.

Define the quasi-Stirling polynomials

$$\overline{Q}_n(t) = \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\operatorname{des}(\pi)}.$$

Example

$\overline{Q}_1(t) = t, \qquad \overline{Q}_2(t) = t + 3t^2, \qquad \overline{Q}_3(t) = t + 13t^2 + 16t^3, \qquad \ldots$

Descents on noncrossing permutations

We are interested not only in how many permutations maximize des, but more generally in the distribution of des over $\overline{\mathcal{Q}}_n$.

Define the quasi-Stirling polynomials

$$\overline{Q}_n(t) = \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\mathrm{des}(\pi)}.$$

Example

$\overline{Q}_1(t) = t, \qquad \overline{Q}_2(t) = t + 3t^2, \qquad \overline{Q}_3(t) = t + 13t^2 + 16t^3, \quad \ldots$

Define their EGF

$$\overline{Q}(t,z) = \sum_{n \geq 0} \overline{Q}_n(t) \frac{z^n}{n!}.$$

Descents on noncrossing permutations

We are interested not only in how many permutations maximize des, but more generally in the distribution of des over $\overline{\mathcal{Q}}_n$.

Define the quasi-Stirling polynomials

$$\overline{Q}_n(t) = \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\operatorname{des}(\pi)}.$$

Example

$\overline{Q}_1(t) = t, \qquad \overline{Q}_2(t) = t + 3t^2, \qquad \overline{Q}_3(t) = t + 13t^2 + 16t^3, \quad \dots$

Define their EGF

$$\overline{Q}(t,z) = \sum_{n \geq 0} \overline{Q}_n(t) \frac{z^n}{n!}.$$

Recall the Eulerian polynomials $A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\operatorname{des}(\pi)}$ and their EGF

$$A(t,z) = \sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{1-te^{(1-t)z}}.$$

Descents on noncrossing permutations

Theorem (E. '21)

The EGF $\overline{Q}(t,z)$ *for noncrossing permutations by the number of descents satisfies the implicit equation*

$$\overline{Q}(t,z) = A(t, z\overline{Q}(t,z)),$$

that is,

$$\overline{Q}(t,z) = \frac{1-t}{1-te^{(1-t)z\overline{Q}(t,z)}}.$$

Descents on noncrossing permutations

Theorem (E. '21)

The EGF $\overline{Q}(t,z)$ *for noncrossing permutations by the number of descents satisfies the implicit equation*

$$\overline{Q}(t,z) = A(t, z\overline{Q}(t,z)),$$

that is,

$$\overline{Q}(t,z) = \frac{1-t}{1-te^{(1-t)z\overline{Q}(t,z)}}.$$

Its coefficients satisfy

$$\overline{Q}_n(t) = \frac{n!}{n+1}\,[z^n]A(t,z)^{n+1}.$$

Here $[z^n]F(z)$ denotes the coefficient of z^n in $F(z)$.

Consequences

Recall the formulas:

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}} \qquad \text{(Eulerian)}$$

Consequences

Recall the formulas:

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}} \qquad \text{(Eulerian)}$$

$$\sum_{m\geq 0} S(m+n,m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}} \qquad \text{(Stirling)}$$

Consequences

Recall the formulas:

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}} \qquad \text{(Eulerian)}$$

$$\sum_{m\geq 0} S(m+n,m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}} \qquad \text{(Stirling)}$$

$$\sum_{m\geq 0} \;??? \;\; t^m = \frac{\overline{Q}_n(t)}{(1-t)^{2n+1}} \qquad \text{(quasi-Stirling)}$$

Consequences

Recall the formulas:

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}} \qquad \text{(Eulerian)}$$

$$\sum_{m\geq 0} S(m+n,m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}} \qquad \text{(Stirling)}$$

Theorem (E. '21)

$$\sum_{m\geq 0} \frac{m^n}{n+1}\binom{m+n}{m} t^m = \frac{\overline{Q}_n(t)}{(1-t)^{2n+1}} \qquad \textit{(quasi-Stirling)}$$

Properties of quasi-Stirling polynomials

Theorem (Bóna '08)

On average, Stirling permutations in $\mathcal{Q}_n$ have $(2n+1)/3$ ascents, $(2n+1)/3$ descents, and $(2n+1)/3$ plateaus.

Properties of quasi-Stirling polynomials

Theorem (Bóna '08)

On average, Stirling permutations in $\mathcal{Q}_n$ have $(2n+1)/3$ ascents, $(2n+1)/3$ descents, and $(2n+1)/3$ plateaus.

Theorem (E. '21)

On average, noncrossing permutations in $\overline{\mathcal{Q}}_n$ have $(3n+1)/4$ ascents, $(3n+1)/4$ descents, and $(n+1)/2$ plateaus.

Properties of quasi-Stirling polynomials

Theorem (Frobenius)

The roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive.

Properties of quasi-Stirling polynomials

Theorem (Frobenius)

The roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive.

Theorem (Brenti '89, Bóna '08)

The same holds for the Stirling polynomials $Q_n(t)$.

Properties of quasi-Stirling polynomials

Theorem (Frobenius)

The roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive.

Theorem (Brenti '89, Bóna '08)

The same holds for the Stirling polynomials $Q_n(t)$.

Theorem (E. '21)

The same holds for the quasi-Stirling polynomials $\overline{Q}_n(t)$.

Properties of quasi-Stirling polynomials

Theorem (Frobenius)

The roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive.

Theorem (Brenti '89, Bóna '08)

The same holds for the Stirling polynomials $Q_n(t)$.

Theorem (E. '21)

The same holds for the quasi-Stirling polynomials $\overline{Q}_n(t)$.

Corollary

- *The coefficients of $\overline{Q}_n(t)$ are unimodal and log-concave.*

Properties of quasi-Stirling polynomials

Theorem (Frobenius)

The roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive.

Theorem (Brenti '89, Bóna '08)

The same holds for the Stirling polynomials $Q_n(t)$.

Theorem (E. '21)

The same holds for the quasi-Stirling polynomials $\overline{Q}_n(t)$.

Corollary

- *The coefficients of $\overline{Q}_n(t)$ are unimodal and log-concave.*
- *The distribution of the number of descents on $\overline{\mathcal{Q}}_n$ is asymptotically normal.*

k-Stirling and k-quasi-Stirling permutations

Gessel and Stanley proposed the following generalization of Stirling permutations by allowing k copies of each element in $\{1, 2, \ldots, n\}$:

k-Stirling and k-quasi-Stirling permutations

Gessel and Stanley proposed the following generalization of Stirling permutations by allowing k copies of each element in $\{1, 2, \ldots, n\}$:

Definition (Gessel–Stanley '78)

A *k-Stirling permutation* is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the pattern 212.

k-Stirling and k-quasi-Stirling permutations

Gessel and Stanley proposed the following generalization of Stirling permutations by allowing k copies of each element in $\{1, 2, \ldots, n\}$:

Definition (Gessel–Stanley '78)

A k-Stirling permutation is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the pattern 212.

$\mathcal{Q}_n^k$ = set of k-Stirling permutations.

k-Stirling and k-quasi-Stirling permutations

Gessel and Stanley proposed the following generalization of Stirling permutations by allowing k copies of each element in $\{1, 2, \dots, n\}$:

Definition (Gessel–Stanley '78)

A *k-Stirling permutation* is a permutation of the multiset $\{1^k, 2^k, \dots, n^k\}$ that avoids the pattern 212.

$\mathcal{Q}_n^k$ = set of k-Stirling permutations.

Also studied by [Brenti '89, Park '94, Janson–Kuba–Panholzer '11].

k-Stirling and k-quasi-Stirling permutations

Gessel and Stanley proposed the following generalization of Stirling permutations by allowing k copies of each element in $\{1, 2, \ldots, n\}$:

Definition (Gessel–Stanley '78)

A *k-Stirling permutation* is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the pattern 212.

$\mathcal{Q}_n^k$ = set of k-Stirling permutations.

Also studied by [Brenti '89, Park '94, Janson–Kuba–Panholzer '11].

Definition

A *k-quasi-Stirling permutation* is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the patterns 1212 and 2121.

k-Stirling and k-quasi-Stirling permutations

Gessel and Stanley proposed the following generalization of Stirling permutations by allowing k copies of each element in $\{1, 2, \ldots, n\}$:

Definition (Gessel–Stanley '78)

A *k-Stirling permutation* is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the pattern 212.

$\mathcal{Q}_n^k$ = set of k-Stirling permutations.

Also studied by [Brenti '89, Park '94, Janson–Kuba–Panholzer '11].

Definition

A *k-quasi-Stirling permutation* is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the patterns 1212 and 2121.

$\overline{\mathcal{Q}}_n^k$ = set of k-quasi-Stirling permutations.

k-Stirling and k-quasi-Stirling permutations

Gessel and Stanley proposed the following generalization of Stirling permutations by allowing k copies of each element in $\{1, 2, \ldots, n\}$:

Definition (Gessel–Stanley '78)

A *k-Stirling permutation* is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the pattern 212.

$\mathcal{Q}_n^k$ = set of k-Stirling permutations.

Also studied by [Brenti '89, Park '94, Janson–Kuba–Panholzer '11].

Definition

A *k-quasi-Stirling permutation* is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the patterns 1212 and 2121.

$\overline{\mathcal{Q}}_n^k$ = set of k-quasi-Stirling permutations.

Note: $\mathcal{Q}_n^1 = \overline{\mathcal{Q}}_n^1 = \mathcal{S}_n$, $\quad \mathcal{Q}_n^2 = \mathcal{Q}_n$, $\quad \overline{\mathcal{Q}}_n^2 = \overline{\mathcal{Q}}_n$.

Generalization to k-Stirling and k-quasi-Stirling

Our results for $k = 2$ generalize to arbitrary k.

Generalization to k-Stirling and k-quasi-Stirling

Our results for $k = 2$ generalize to arbitrary k.

To obtain them, we generalize φ to a bijection between k-quasi-Stirling permutations and certain trees.

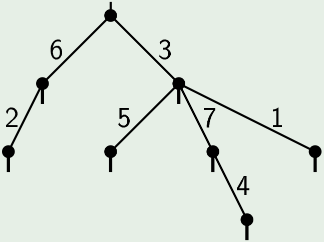
Generalization to k-Stirling and k-quasi-Stirling

Our results for $k = 2$ generalize to arbitrary k.

To obtain them, we generalize φ to a bijection between k-quasi-Stirling permutations and certain trees.

Example

A bijection between *compartmented trees* and 3-quasi-Stirling permutations:



$\xrightarrow{\phi}$ 622266355537744471113

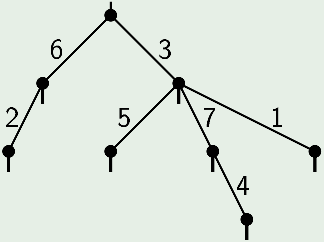
Generalization to k-Stirling and k-quasi-Stirling

Our results for $k = 2$ generalize to arbitrary k.

To obtain them, we generalize φ to a bijection between k-quasi-Stirling permutations and certain trees.

Example

A bijection between *compartmented trees* and 3-quasi-Stirling permutations:



$$\xrightarrow{\phi} \quad 6222663555377444771113$$

Additionally, we can add a variable to the generating functions that keeps track of the number of plateaus.

Nonnesting permutations

Nonnesting permutations

Definition

A nonnesting permutation (or canon permutation) is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1221 and 2112.

Nonnesting permutations

Definition

A nonnesting permutation (or canon permutation) is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1221 and 2112.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_\ell$ and $\pi_j = \pi_k$.

Nonnesting permutations

Definition

A nonnesting permutation (or canon permutation) is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1221 and 2112.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_\ell$ and $\pi_j = \pi_k$.

$\mathcal{C}_n$ = set of nonnesting permutations of $[n] \sqcup [n]$.

Nonnesting permutations

Definition

A nonnesting permutation (or canon permutation) is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1221 and 2112.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_\ell$ and $\pi_j = \pi_k$.

$\mathcal{C}_n$ = set of nonnesting permutations of $[n] \sqcup [n]$.

Example

$3532521414 \in \mathcal{C}_5$,

Nonnesting permutations

Definition

A nonnesting permutation (or canon permutation) is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1221 and 2112.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_\ell$ and $\pi_j = \pi_k$.

$\mathcal{C}_n$ = set of nonnesting permutations of $[n] \sqcup [n]$.

Example

$3532521414 \in \mathcal{C}_5$, 312321

Nonnesting permutations

Definition

A nonnesting permutation (or canon permutation) is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1221 and 2112.

Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_\ell$ and $\pi_j = \pi_k$.

$\mathcal{C}_n$ = set of nonnesting permutations of $[n] \sqcup [n]$.

Example

$3532521414 \in \mathcal{C}_5, \qquad 312321 \notin \mathcal{C}_3.$

Nonnesting permutations

Definition

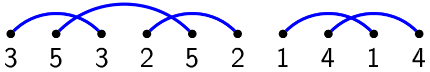
A nonnesting permutation (or canon permutation) is a permutation of the multiset $[n] \sqcup [n]$ that avoids the patterns 1221 and 2112.

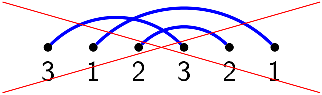
Equivalently, there do not exist $i < j < k < \ell$ with $\pi_i = \pi_\ell$ and $\pi_j = \pi_k$.

$\mathcal{C}_n$ = set of nonnesting permutations of $[n] \sqcup [n]$.

Example

$3532521414 \in \mathcal{C}_5$, $312321 \notin \mathcal{C}_3$.





They are in bijection with labeled nonnesting matchings, so again

$$|\mathcal{C}_n| = n!\mathrm{Cat}_n = \frac{(2n)!}{(n+1)!}.$$

Nonnesting permutations

A permutation π of $[n] \sqcup [n]$ is nonnesting iff the subsequence of first copies of each entry coincides with the subsequence of second copies.

Example

$\pi = 3532521414 \in \mathcal{C}_5$, .

Nonnesting permutations

A permutation π of $[n] \sqcup [n]$ is nonnesting iff the subsequence of first copies of each entry coincides with the subsequence of second copies.

We denote this subsequence by $s(\pi)$.

Example

$\pi = 3532521414 \in \mathcal{C}_5, \quad s(\pi) = 35214 \in \mathcal{S}_5.$

Nonnesting permutations

A permutation π of $[n] \sqcup [n]$ is nonnesting iff the subsequence of first copies of each entry coincides with the subsequence of second copies.

We denote this subsequence by $s(\pi)$.

Example

$\pi = 3532521414 \in \mathcal{C}_5, \quad s(\pi) = 35214 \in \mathcal{S}_5.$

Our goal is to count nonnesting permutations with respect to the number of descents and plateaus. Consider the polynomials

$$C_n(t, u) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)}.$$

Nonnesting permutations

A permutation π of $[n] \sqcup [n]$ is nonnesting iff the subsequence of first copies of each entry coincides with the subsequence of second copies.

We denote this subsequence by $s(\pi)$.

Example

$\pi = 3532521414 \in \mathcal{C}_5, \quad s(\pi) = 35214 \in \mathcal{S}_5.$

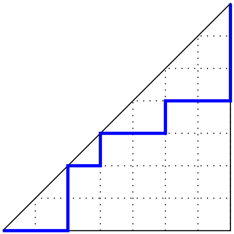
Our goal is to count nonnesting permutations with respect to the number of descents and plateaus. Consider the polynomials

$$C_n(t,u) = \sum_{\pi \in \mathcal{C}_n} t^{\operatorname{des}(\pi)} u^{\operatorname{plat}(\pi)}.$$

Even though $|\mathcal{C}_n| = |\overline{\mathcal{Q}}_n|$, we have $\displaystyle\sum_{\pi \in \mathcal{C}_n} t^{\operatorname{des}(\pi)} \neq \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\operatorname{des}(\pi)}$.

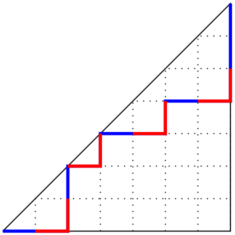
Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathtt{e} = (1,0)$ and $\mathtt{n} = (0,1)$ that do not go above the diagonal $y = x$.



Dyck paths and Narayana numbers

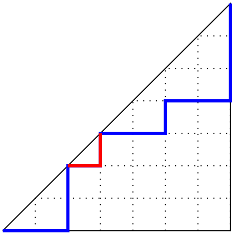
Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathtt{e}=(1,0)$ and $\mathtt{n}=(0,1)$ that do not go above the diagonal $y=x$.



A *peak* of $D \in \mathcal{D}_n$ is an occurrence of $\mathtt{en}$.

Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathrm{e} = (1,0)$ and $\mathtt{n} = (0,1)$ that do not go above the diagonal $y = x$.

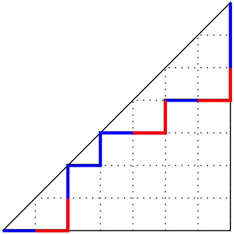


A *peak* of $D \in \mathcal{D}_n$ is an occurrence of $\mathtt{en}$.

A peak is called a *low peak* if it touches the diagonal,

Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathtt{e} = (1,0)$ and $\mathtt{n} = (0,1)$ that do not go above the diagonal $y = x$.

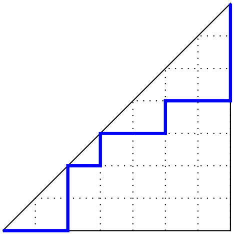


A *peak* of $D \in \mathcal{D}_n$ is an occurrence of $\mathtt{en}$.

A peak is called a *low peak* if it touches the diagonal, and a *high peak* otherwise.

Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathrm{e} = (1,0)$ and $\mathtt{n} = (0,1)$ that do not go above the diagonal $y = x$.



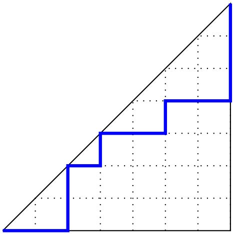
A *peak* of $D \in \mathcal{D}_n$ is an occurrence of $\mathtt{en}$.

A peak is called a *low peak* if it touches the diagonal, and a *high peak* otherwise.

Let $\mathrm{lpea}(D)$ and $\mathrm{hpea}(D)$ denote the number of low peaks and high peaks of D, respectively.

Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathtt{e}=(1,0)$ and $\mathtt{n}=(0,1)$ that do not go above the diagonal $y=x$.



A *peak* of $D\in\mathcal{D}_n$ is an occurrence of $\mathtt{en}$.

A peak is called a *low peak* if it touches the diagonal, and a *high peak* otherwise.

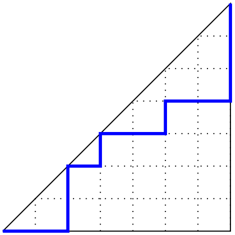
Let $\mathrm{lpea}(D)$ and $\mathrm{hpea}(D)$ denote the number of low peaks and high peaks of D, respectively.

Let

$$N_n(t,u)=\sum_{D\in\mathcal{D}_n} t^{\mathrm{hpea}(D)}u^{\mathrm{lpea}(D)}.$$

Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathtt{e} = (1,0)$ and $\mathtt{n} = (0,1)$ that do not go above the diagonal $y = x$.



A *peak* of $D \in \mathcal{D}_n$ is an occurrence of $\mathtt{en}$.

A peak is called a *low peak* if it touches the diagonal, and a *high peak* otherwise.

Let $\mathrm{lpea}(D)$ and $\mathrm{hpea}(D)$ denote the number of low peaks and high peaks of D, respectively.

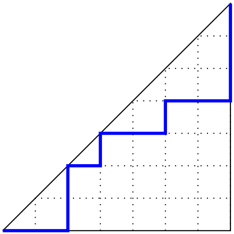
Let

$$N_n(t,u) = \sum_{D \in \mathcal{D}_n} t^{\mathrm{hpea}(D)} u^{\mathrm{lpea}(D)}.$$

The coefficients of $N_n(t,t)$ are the *Narayana numbers* $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathtt{e} = (1,0)$ and $\mathtt{n} = (0,1)$ that do not go above the diagonal $y = x$.



A *peak* of $D \in \mathcal{D}_n$ is an occurrence of $\mathtt{en}$.

A peak is called a *low peak* if it touches the diagonal, and a *high peak* otherwise.

Let $\mathrm{lpea}(D)$ and $\mathrm{hpea}(D)$ denote the number of low peaks and high peaks of D, respectively.

Let

$$N_n(t,u) = \sum_{D \in \mathcal{D}_n} t^{\mathrm{hpea}(D)} u^{\mathrm{lpea}(D)}.$$

The coefficients of $N_n(t,t)$ are the *Narayana numbers* $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

$$\sum_{n \geq 0} N_n(t,u) z^n = \frac{2}{1 + (1 + t - 2u)z + \sqrt{1 - 2(1+t)z + (1-t)^2 z^2}}.$$

Descents and plateaus on nonnesting permutations

Recall:

$$C_n(t,u) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)},$$

$$A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{des}(\pi)},$$

$$N_n(t,u) = \sum_{D \in \mathcal{D}_n} t^{\mathrm{hpea}(D)} u^{\mathrm{lpea}(D)}.$$

Descents and plateaus on nonnesting permutations

Recall:

$$C_n(t,u) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)},$$

$$A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{des}(\pi)},$$

$$N_n(t,u) = \sum_{D \in \mathcal{D}_n} t^{\mathrm{hpea}(D)} u^{\mathrm{lpea}(D)}.$$

Theorem (E. '22)

$$C_n(t,u) = A_n(t)\, N_n(t,u).$$

Descents and plateaus on nonnesting permutations

Recall:

$$C_n(t,u) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)},$$

$$A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{des}(\pi)},$$

$$N_n(t,u) = \sum_{D \in \mathcal{D}_n} t^{\mathrm{hpea}(D)} u^{\mathrm{lpea}(D)}.$$

Theorem (E. '22)

$$C_n(t,u) = A_n(t)\, N_n(t,u).$$

Example

$$\begin{aligned} C_3(t,u) &= u^3 t + (1+2u+4u^3)t^2 + (5+8u+u^3)t^3 + (5+2u)t^4 + t^5 \\ &= \left(t+4t^2+t^3\right)\left(u^3+(1+2u)t+t^2\right). \end{aligned}$$

Consequences

Since both $A_n(t)$ and $N_n(t,t)$ are palindromic, so is their product $C_n(t,t)$.

Example

$$C_3(t,t) = t^2 + 7t^3 + 14t^4 + 7t^5 + t^6 = (t + 4t^2 + t^3)(t + 3t^2 + t^3).$$

Consequences

Since both $A_n(t)$ and $N_n(t,t)$ are palindromic, so is their product $C_n(t,t)$.

Example

$$C_3(t,t) = t^2 + 7t^3 + 14t^4 + 7t^5 + t^6 = (t + 4t^2 + t^3)(t + 3t^2 + t^3).$$

Note that

$$C_n(t,t) = \sum_{\pi \in \mathcal{C}_n} t^{\operatorname{des}(\pi)} t^{\operatorname{plat}(\pi)} = \sum_{\pi \in \mathcal{C}_n} t^{\operatorname{wdes}(\pi)},$$

where $\operatorname{wdes}(\pi) = \operatorname{des}(\pi) + \operatorname{plat}(\pi)$ is the number of weak descents of π.

Consequences

Since both $A_n(t)$ and $N_n(t,t)$ are palindromic, so is their product $C_n(t,t)$.

Example

$$C_3(t,t) = t^2 + 7t^3 + 14t^4 + 7t^5 + t^6 = (t + 4t^2 + t^3)(t + 3t^2 + t^3).$$

Note that

$$C_n(t,t) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{des}(\pi)} t^{\mathrm{plat}(\pi)} = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{wdes}(\pi)},$$

where $\mathrm{wdes}(\pi) = \mathrm{des}(\pi) + \mathrm{plat}(\pi)$ is the number of weak descents of π.

Corollary

The distribution of weak descents on $\mathcal{C}_n$ is symmetric: for all r,

$$|\{\pi \in \mathcal{C}_n : \mathrm{wdes}(\pi) = r\}| = |\{\pi \in \mathcal{C}_n : \mathrm{wdes}(\pi) = 2n + 2 - r\}|.$$

Consequences

Similarly, since $N_n(t,1)$ is palindromic, so is $A_n(t)N_n(t,1) = C_n(t,1)$.

Example

$$C_3(t,1) = 1 + 7t + 14t^2 + 7t^3 + t^4 = (1+4t+t^2)(1+3t+t^2).$$

Consequences

Similarly, since $N_n(t,1)$ is palindromic, so is $A_n(t)N_n(t,1) = C_n(t,1)$.

Example

$$C_3(t,1) = 1 + 7t + 14t^2 + 7t^3 + t^4 = (1 + 4t + t^2)(1 + 3t + t^2).$$

Corollary

The distribution of descents on $\mathcal{C}_n$ is symmetric: for all r,

$$|\{\pi \in \mathcal{C}_n : \mathrm{des}(\pi) = r\}| = |\{\pi \in \mathcal{C}_n : \mathrm{des}(\pi) = 2n - r\}|.$$

Consequences

Similarly, since $N_n(t,1)$ is palindromic, so is $A_n(t)N_n(t,1) = C_n(t,1)$.

Example

$$C_3(t,1) = 1 + 7t + 14t^2 + 7t^3 + t^4 = (1 + 4t + t^2)(1 + 3t + t^2).$$

Corollary

The distribution of descents on $\mathcal{C}_n$ is symmetric: for all r,

$$|\{\pi \in \mathcal{C}_n : \mathrm{des}(\pi) = r\}| = |\{\pi \in \mathcal{C}_n : \mathrm{des}(\pi) = 2n - r\}|.$$

We have bijective proofs of these corollaries but they are surprisingly complicated!

A refinement

Partition the set $\mathcal{C}_n$ according to the permutation $\sigma \in \mathcal{S}_n$ given by the first copy of each entry:

$$\mathcal{C}_n^\sigma = \{\pi \in \mathcal{C}_n : s(\pi) = \sigma\}$$

A refinement

Partition the set $\mathcal{C}_n$ according to the permutation $\sigma \in \mathcal{S}_n$ given by the first copy of each entry:

$$\mathcal{C}_n^\sigma = \{\pi \in \mathcal{C}_n : s(\pi) = \sigma\}$$

Example

$3532521414 \in \mathcal{C}_5^{35214}$

A refinement

Partition the set $\mathcal{C}_n$ according to the permutation $\sigma \in \mathcal{S}_n$ given by the first copy of each entry:

$$\mathcal{C}_n^\sigma = \{\pi \in \mathcal{C}_n : s(\pi) = \sigma\}$$

Example

$3532521414 \in \mathcal{C}_5^{35214}$

Let

$$C_n^\sigma(t,u) = \sum_{\pi \in \mathcal{C}_n^\sigma} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)}.$$

A refinement

Partition the set $\mathcal{C}_n$ according to the permutation $\sigma \in \mathcal{S}_n$ given by the first copy of each entry:

$$\mathcal{C}_n^\sigma = \{\pi \in \mathcal{C}_n : s(\pi) = \sigma\}$$

Example

$3532521414 \in \mathcal{C}_5^{35214}$

Let

$$C_n^\sigma(t,u) = \sum_{\pi \in \mathcal{C}_n^\sigma} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)}.$$

Theorem (E. '22)

For all $\sigma \in \mathcal{S}_n$,

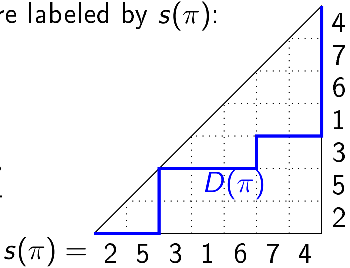
$$C_n^\sigma(t,u) = t^{\mathrm{des}(\sigma)} N_n(t,u).$$

Summing over $\sigma \in \mathcal{S}_n$, we obtain the previous theorem.

About the proofs

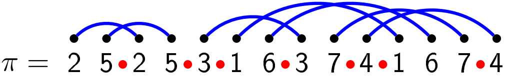
Using the standard bijection between nonnesting matchings and Dyck paths, we can represent a nonnesting permutation $\pi \in \mathcal{C}_n$ as a Dyck path $D(\pi)$ in a grid whose rows and columns are labeled by $s(\pi)$:

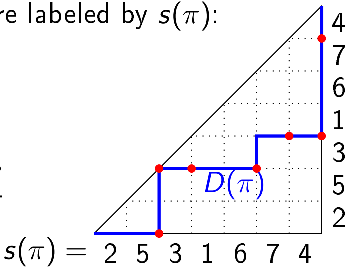
$\pi = 2\ 5\ 2\ 5\ 3\ 1\ 6\ 3\ 7\ 4\ 1\ 6\ 7\ 4$



About the proofs

Using the standard bijection between nonnesting matchings and Dyck paths, we can represent a nonnesting permutation $\pi \in \mathcal{C}_n$ as a Dyck path $D(\pi)$ in a grid whose rows and columns are labeled by $s(\pi)$:

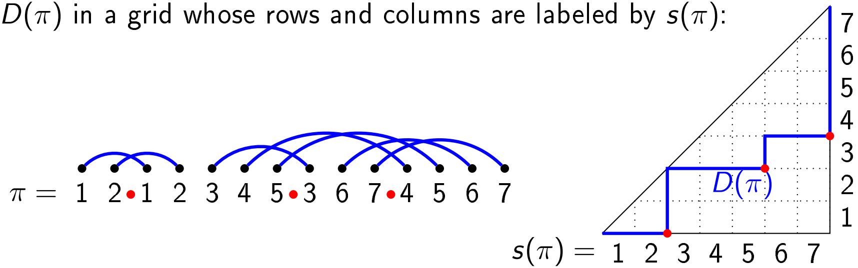




Plateaus of π correspond to low peaks of $D(\pi)$.
But what do descents correspond to?

About the proofs

Using the standard bijection between nonnesting matchings and Dyck paths, we can represent a nonnesting permutation $\pi \in \mathcal{C}_n$ as a Dyck path $D(\pi)$ in a grid whose rows and columns are labeled by $s(\pi)$:

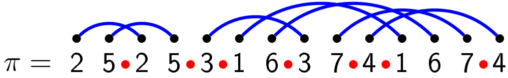


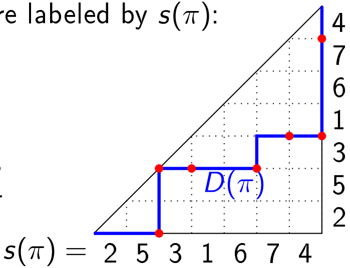
Plateaus of π correspond to low peaks of $D(\pi)$.
But what do descents correspond to?

In the special case that $s(\pi) = 12\dots n$, descents of π correspond to high peaks of $D(\pi)$, proving that $C_n^{12\dots n}(t,u) = tN_n(t,u)$.

About the proofs

Using the standard bijection between nonnesting matchings and Dyck paths, we can represent a nonnesting permutation $\pi \in \mathcal{C}_n$ as a Dyck path $D(\pi)$ in a grid whose rows and columns are labeled by $s(\pi)$:





Plateaus of π correspond to low peaks of $D(\pi)$.
But what do descents correspond to?

In the special case that $s(\pi) = 12\dots n$, descents of π correspond to high peaks of $D(\pi)$, proving that $C_n^{12\dots n}(t,u) = tN_n(t,u)$.

In general, for each fixed $\sigma \in \mathcal{S}_n$, we get a different Dyck path statistic. We prove that they all have a (shifted) Narayana distribution.

Generalizations

Our theorem generalizes to permutations that have k copies of each number in $[n]$, for any given k.

Generalizations

Our theorem generalizes to permutations that have k copies of each number in $[n]$, for any given k.

However, there is more than one way to generalize the definition of nonnesting.

Generalizations

Our theorem generalizes to permutations that have k copies of each number in $[n]$, for any given k.

However, there is more than one way to generalize the definition of nonnesting.

Instead of requiring avoidance of 1221 and 2112, the "correct" generalization is the one that arises from the *canon* interpretation.

Example

353325215241414

Generalizations

Our theorem generalizes to permutations that have k copies of each number in $[n]$, for any given k.

However, there is more than one way to generalize the definition of nonnesting.

Instead of requiring avoidance of 1221 and 2112, the "correct" generalization is the one that arises from the *canon* interpretation.

Example

353325215241414

In the proof for the general case, the role of Dyck paths is played by standard Young tableaux of rectangular shape.

Thank you