Bijections for derangements and pattern-avoiding inversion sequences

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Notation for derangements

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$\mathcal{D}_n = \{\pi \in \mathcal{S}_n : \pi$ has no fixed points$\}$ $\qquad d_n = |\mathcal{D}_n|$

Non-derangements:

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Permutations with one fixed point:

$\mathcal{F}_n = \{\pi \in \mathcal{S}_n : \pi \text{ has exactly one fixed point}\}$ $\qquad |\mathcal{F}_n| = n\, d_{n-1}$

Known recurrences for the derangement numbers

Recurrence 1: For $n \geq 2$,

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Here we present a new bijective proof of Recurrence 2 that is arguably simpler than these.

A bijective proof of $d_n = n\,d_{n-1} + (-1)^n$

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We describe a bijection

$$\psi : \mathcal{D}_n^* \to \mathcal{F}_n^*,$$

where

$$\mathcal{D}_n^* = \begin{cases} \mathcal{D}_n \setminus \{(1,2)(3,4)\dots(n-1,n)\} & \text{if } n \text{ even}, \\ \mathcal{D}_n & \text{if } n \text{ odd}, \end{cases}$$

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$$\pi = (1,2)(3,4)\dots(2k-1,2k)\square$$

To define $\psi(\pi) \in \mathcal{F}_n^*$, consider two cases:

1. If the cycle containing $2k+1$ has at least 3 elements:

$$\begin{aligned} \pi &= (1,2)(3,4)\dots(2k-1,2k)(2k+1,a_1,a_2,\dots,a_j)\square \\ \psi(\pi) &= (1)(2,3)(4,5) \quad \dots \quad (2k,a_1)(2k+1,a_2,\dots,a_j)\square \end{aligned}$$

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2. Otherwise:

$$\pi = (1,2)(3,4)\dots(2k-1,2k)(2k+1, a_1)(2k+2, a_2, \dots, a_j)\square$$
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Examples of $\psi : \mathcal{D}_n^* \to \mathcal{F}_n^*$

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Examples:

π	(13)(24)	(14)(23)	(1234)	(1243)	(1342)	(1423)	...
$\psi(\pi)$	(1)(234)	(1)(243)	(2)(134)	(2)(143)	(3)(142)	(4)(123)	...

π	(12)(345)	(123)(45)	(13)(254)	(14)(235)	(154)(23)	...
$\psi(\pi)$	(1)(24)(35)	(2)(13)(45)	(1)(2354)	(1)(2435)	(5)(14)(23)	...

Inversion sequences

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Theorem (Auli, E. '19)

$$|\mathbf{I}_n(\underline{000})| = \frac{(n+1)! - d_{n+1}}{n}.$$

The original proof was by induction on n.
Here we provide a bijective proof.

A bijective proof of $|\mathbf{I}_n(\underline{000})| = \frac{(n+1)!-d_{n+1}}{n}$

One can easily show that

$$\frac{(n+1)! - d_{n+1}}{n} = |\overline{\mathcal{D}}_n \sqcup \overline{\mathcal{D}}_{n-1}|,$$

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Step 1: Encode $e \in \mathbf{I}_n(\underline{000})$ as a word $w = w_2 \dots w_n$ with $w_k \in [k-1] \cup \{R\}$ having no two consecutive Rs, by letting

$$w_k = \begin{cases} R & \text{if } e_k = e_{k-1}, \\ e_k & \text{if } e_k > e_{k-1}, \\ e_k + 1 & \text{if } e_k < e_{k-1}. \end{cases}$$

The bijection $\phi : \mathbf{I}_n(\underline{000}) \to \overline{\mathcal{D}}_n \sqcup \overline{\mathcal{D}}_{n-1}$

Step 2: Read w from left to right and build a sequence of permutations $\sigma_1, \sigma_2, \ldots, \sigma_n$, where $\sigma_k \in \overline{\mathcal{D}}_k \sqcup \overline{\mathcal{D}}_{k-1}$ for all k.

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Set $\sigma_1 = 1 \in \overline{\mathcal{D}}_1$. Then, for each k from 2 to n:

- If $w_k = R$, let $\sigma_k = \sigma_{k-1} \in \overline{\mathcal{D}}_{k-1}$.

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- Otherwise, let

$$\sigma_k = \begin{cases} (w_k, k)\sigma_{k-1} & \text{if } w_{k-1} \neq R \text{ and } \sigma_{k-1} \in \overline{\mathcal{D}}_{k-1} \text{ has} \\ & \text{fixed points other than } w_k, \\ (w_k, k-1)\sigma_{k-1} & \text{otherwise,} \end{cases}$$

  where $(a, b)\sigma_{k-1}$ is defined by viewing σ_{k-1} as an element of $\overline{\mathcal{D}}_k$ (where k is fixed), and switching the entries a and b.

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Finally, let $\phi(e) = \sigma_n$.

Examples of $\phi : \mathbf{I}_n(\underline{000}) \to \overline{\mathcal{D}}_n \sqcup \overline{\mathcal{D}}_{n-1}$

$e = 001322 \quad \mapsto$

k	e_k	w_k	σ_k
1	0		1
2	0	R	1
3	1	1	$(1,2)123 = 213$
4	3	3	$(3,3)2134 = 2134$
5	2	3	$(3,5)21345 = 21543$
6	2	R	$21543 = \phi(e)$

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5	2	3	$(3,5)21345 = 21543$
6	2	R	$21543 = \phi(e)$

$e = 0102230 \quad \mapsto$

k	e_k	w_k	σ_k
1	0		1
2	1	1	$(1,1)12 = 12$
3	0	1	$(1,3)123 = 321$
4	2	2	$(2,3)3214 = 2314$
5	2	R	2314
6	3	3	$(3,5)231456 = 251436$
7	0	1	$(1,7)2514367 = 2574361 = \phi(e)$

Thank you