Refined enumeration of pattern-avoiding permutations

Sergi Elizalde

MSRI

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http://www.mrsi.org/∼elizalde

Overview

- Pattern avoidance
  - Definitions
  - Exact enumeration
  - Asymptotic enumeration
  - Simultaneous avoidance
  - Consecutive patterns
  - Generalized patterns

- Statistics on pattern-avoiding permutations
  - Equidistribution results of fixed points and excedances
  - Statistics on Dyck paths
  - Bijective proofs
  - Statistics and simultaneous avoidance
  - Statistics and generalized patterns

Definitions

$\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_k$

π *contains* σ if there exist $i_1 < \ldots < i_k$ s.t. $\pi_{i_1}\cdots\pi_{i_k}$ has type σ (i.e., $\pi_{i_a} < \pi_{i_b} \Longleftrightarrow \sigma_a < \sigma_b$).

Otherwise, we say that π *avoids* σ.

Example:
2453 1 contains 132
42351 avoids 132

$\mathcal{S}_n(\sigma) := \{\pi \in \mathcal{S}_n : \pi \text{ avoids } \sigma\}$

Basic questions:

- What can we say about $|\mathcal{S}_n(\sigma)|$? Exact formula? Asymptotic formula?
- For which σ_1 and σ_2 do we have $|\mathcal{S}_n(\sigma_1)| = |\mathcal{S}_n(\sigma_2)|$?

Patterns of length 3

By trivial bijections,

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(321)|$$

$$|\mathcal{S}_n(132)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(312)| = |\mathcal{S}_n(213)|$$

Knuth '73.

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(132)| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

(Catalan number)

Patterns of length 4

West '90, Stankova '94, '96.

Patterns $\sigma \in \mathcal{S}_4$ fall in three different classes:

(a) 1234 $\longrightarrow$ enumerated by I. Gessel

(b) 1342 $\longrightarrow$ enumerated by M. Bóna

(c) 1324 $\longrightarrow$ no formula known for $|\mathcal{S}_n(1324)|$

Bóna '97. For $n \geq 7$,

$$|\mathcal{S}_n(1342)| < |\mathcal{S}_n(1234)| < |\mathcal{S}_n(1324)|$$

Bóna '97.

$$\sum_{n\geq 0} |\mathcal{S}_n(1342)| z^n = \frac{32z}{1 + 20z - 8z^2 - (1-8z)^{3/2}}$$

Proof: bijection between *indecomposable* 1324-avoiding permutations and a certain kind of labelled trees.

Patterns of arbitrary length

Gessel '90.

$$\sum_{n\geq 0} \frac{|\mathcal{S}_n(123\cdots k)|}{n!^2} z^{2n} = \det(I_{|r-s|}(2z))_{r,s=1,\ldots,k-1}$$

where $I_j(2z) = \sum_{n\geq 0} \frac{z^{2n+j}}{n!(n-j)!}$ are Bessel functions of imaginary argument.

Babson, West, Backelin, Xin '01. For all r, t, n,

$$|\mathcal{S}_n(123\cdots r a_{r+1} a_{r+2} \cdots a_{r+t})| = |\mathcal{S}_n(r\cdots 321 a_{r+1} a_{r+2} \cdots a_{r+t})|$$

Asymptotic enumeration

Regev '81.

$$|S_n(123\cdots k)| \sim c\,\frac{(k-1)^{2n}}{n^{\frac{k^2-2k}{2}}}$$

Stanley-Wilf Conjecture '90.
For every pattern σ, there exists λ s.t.

$$|\mathcal{S}_n(\sigma)| < \lambda^n$$

for all n.

Proved very recently by **Marcus** and **Tardos**.
Idea of the proof:

- Generalize avoidance to 0-1 matrices
- For a 0-1 permutation matrix P, let
  $f(n,P) :=$ max # of 1's in an $n\times n$ 0-1 matrix
  avoiding P
- Main result: $f(n,P) = O(n)$ if P permutation matrix
- The theorem follows from a result of Klazar

Simultaneous avoidance

Defn:

$$\mathcal{S}_n(\sigma_1, \ldots, \sigma_m) = \bigcap_{i=1}^{m} \mathcal{S}_n(\sigma_i)$$

Simion, Schmidt '85. Formula for $|\mathcal{S}_n(\Sigma)|$ for every $\Sigma \subseteq \mathcal{S}_3$. Examples:

$|\mathcal{S}_n(123, 132)| = 2^{n-1}$

$|\mathcal{S}_n(132, 321)| = \binom{n}{2} + 1$

$|\mathcal{S}_n(123, 132, 213)| = F_{n+1}$ (Fibonacci number)

$|\mathcal{S}_n(123, 132, 231)| = n$

West '96. Formulas for $|\mathcal{S}_n(\sigma_1, \sigma_2)|$ where $\sigma_1 \in \mathcal{S}_3$, $\sigma_2 \in \mathcal{S}_4$. Examples:

$|\mathcal{S}_n(123, 3241)| = 3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$

$|\mathcal{S}_n(123, 3214)| = F_{2n}$

Proof uses generating trees.

Gire '93, Kremer '00, West '96.

$|\mathcal{S}_n(\sigma_1, \sigma_2)| = r_{n-1}$ (large Schröder number)

for several pairs $\sigma_1, \sigma_2 \in \mathcal{S}_4$

Consecutive patterns

Defn:

π *contains* the *consecutive pattern* σ if $\exists\ i$ s.t. $\pi_{i+1} \cdots \pi_{i+k}$ has type σ.

Example:

1<u>352</u>4 contains 231

23541 avoids 231

$$c_\sigma(\pi) := \#\text{ occurrences of } \sigma \text{ in } \pi$$

$$P_\sigma(u,z) := \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n} u^{c_\sigma(\pi)} \frac{z^n}{n!}$$

E., Noy '00.

a)

$$\sigma = 12\cdots(m+2)$$

$P_\sigma(u,z) = \dfrac{1}{\omega(u,z)}$, where ω is the solution of

$$\omega^{(m+1)} + (1-u)(\omega^{(m)} + \omega^{(m-1)} + \ldots + \omega' + \omega) = 0$$

$\omega(0) = 1,\ \omega'(0) = -1,\ \omega^{(k)}(0) = 0,\ 2 \le k \le m.$

b)

$$\sigma = 12\cdots(a-1)a\underbrace{\boxed{}}_{\text{any perm. of } \{a+2,\, a+3,\, \ldots,\, m+2\}}(a+1)$$

$P_\sigma(u,z) = \dfrac{1}{\xi(u,z)}$, where ξ is the solution of

$$\xi^{(a+1)} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}\xi' = 0$$

$\xi(0) = 1,\ \xi'(0) = -1,\ \xi^{(k)}(0) = 0,\ 2 \le k \le a.$

Proof uses representations of permutations as increasing binary trees.

Generalized patterns

(Babson, Steingrímsson '00)

Dashes between some letters of σ.
If no dash, elements have to be adjacent in π.
Example:
$\underline{35}42\underline{7}1\underline{6}$ contains 12-4-3, but it avoids 12-43.

They generalize both classical patterns and consecutive patterns.

Claesson '01.

$|\mathcal{S}_n(1\text{-}23)| = |\mathcal{S}_n(1\text{-}32)| = B_n$ (Bell number)

$|\mathcal{S}_n(2\text{-}13)| = C_n$

$|\mathcal{S}_n(1\text{-}23, 12\text{-}3)| = B_n^*$ (Bessel number)

$|\mathcal{S}_n(1\text{-}23, 1\text{-}32)| = I_n$ (involutions in $\mathcal{S}_n$)

$|\mathcal{S}_n(1\text{-}23, 13\text{-}2)| = M_n$ (Motzkin number)

Permutation statistics

π_i is a *fixed point* of π if $\pi_i = i$
$\mathrm{fp}(\pi) :=$ number of fixed points of π

π_i is an *excedance* of π if $\pi_i > i$
$\mathrm{exc}(\pi) :=$ number of excedances of π

π_i is a *descent* of π if $\pi_i > \pi_{i+1}$
(otw. π_i is a *rise*)
$\mathrm{des}(\pi) :=$ number of descents of π

$\mathrm{lis}(\pi) :=$ length of longest increasing subseq. of π

$\mathrm{lds}(\pi) :=$ length of longest decreasing subseq. of π

Ex: if $\pi = 4{\cdot}2{\cdot}1\,7{\cdot}5{\cdot}3\,6$, then
$\mathrm{fp}(\pi) = 2$, $\mathrm{exc}(\pi) = 2$, $\mathrm{des}(\pi) = 4$,
$\mathrm{lis}(\pi) = 3$, $\mathrm{lds}(\pi) = 3$

Robertson, Saracino, Zeilberger '02.
For any k, n,

$$|\{\pi \in \mathcal{S}_n(321) : \text{fp}(\pi) = k\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \text{fp}(\pi) = k\}|$$

Their proof is not bijective.

Questions:

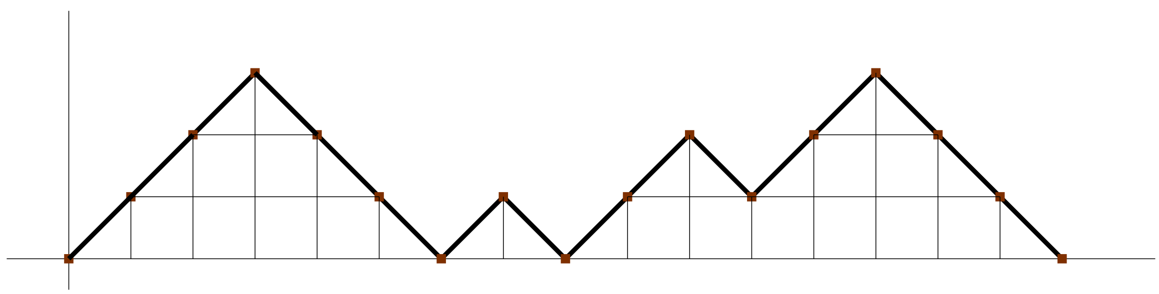
- Is there a simple bijective proof?
- Can this theorem be generalized, considering other statistics in permutations?

We will see a bijective proof.
Idea: bijections between restricted permutations and Dyck paths.

Dyck paths and Motzkin paths

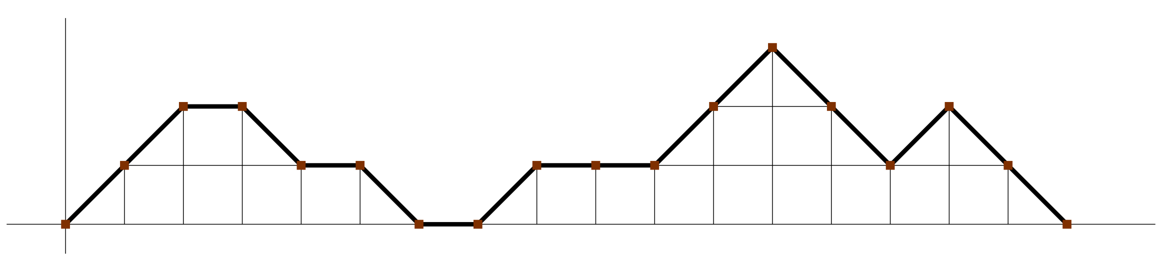
Start and end at the x-axis; never go below it.

Dyck path: steps $u = (1, 1)$ and $d = (1, -1)$.



$\mathcal{D}_n :=$ set of Dyck paths of length $2n$
$|\mathcal{D}_n| = C_n$

Motzkin path: steps $u = (1, 1)$, $d = (1, -1)$ and $h = (1, 0)$.



$\mathcal{M}_n :=$ set of Motzkin paths of length n
$|\mathcal{M}_n| = M_n$

For a Dyck path D, define:

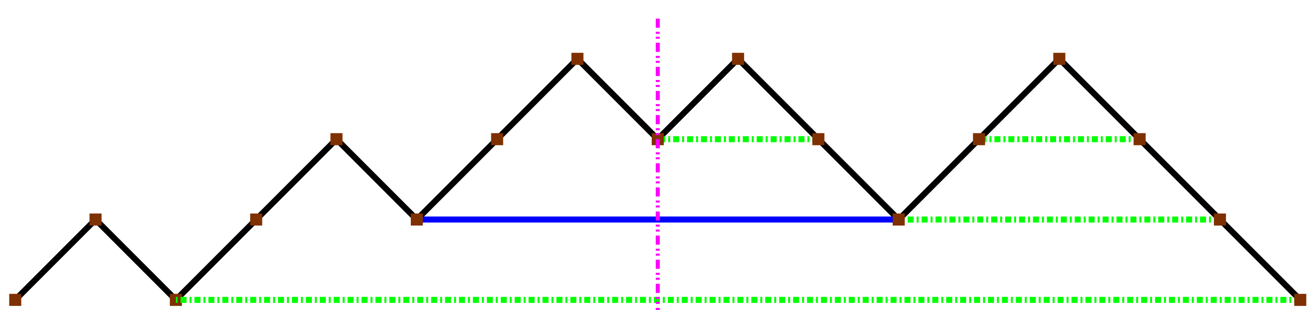
peak: ud (up-step followed by down-step)

hill: peak at height 1

tunnel: horizontal segment between two lattice points of D that stays always below D
(D has n tunnels, one for each step u)

centered tunnel: x-coordinate of midpoint is at the middle of D

right tunnel: x-coordinate of midpoint is in the right half of D



A simple bijective proof of

$$|\{\pi \in \mathcal{S}_n(321) : \text{fp}(\pi) = k\}| = |\{\pi \in \mathcal{S}_n(132) : \text{fp}(\pi) = k\}|$$

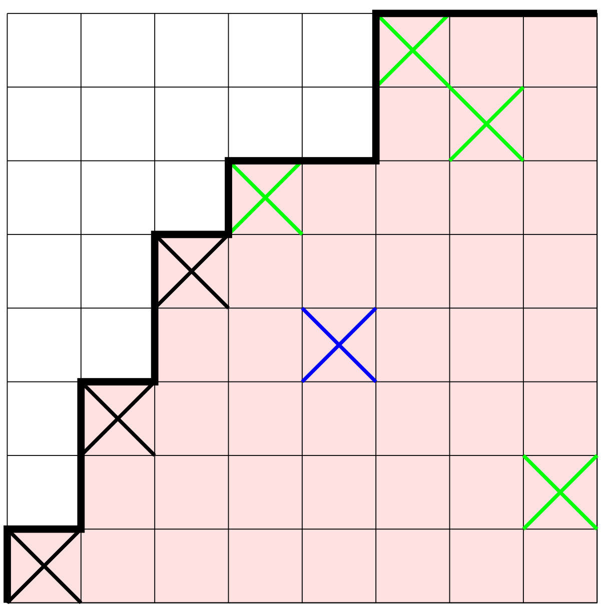
Composition of bijections:

$$\mathcal{S}_n(132) \overset{\varphi_{\text{Krat}}}{\longleftrightarrow} \mathcal{D}_n \overset{\Phi_{\text{ED}}}{\longleftrightarrow} \mathcal{D}_n \overset{\psi}{\longleftrightarrow} S_n(321)$$

fixed points $\leftrightarrow$ centered tunnels $\leftrightarrow$ hills $\leftrightarrow$ fixed points

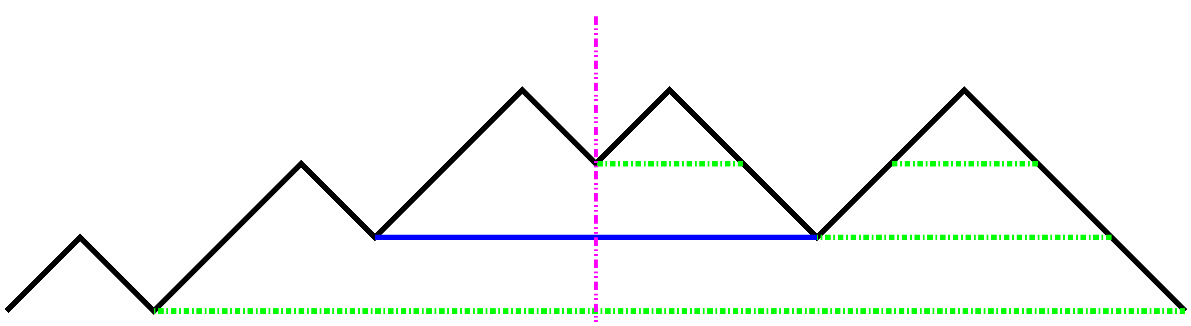
$\varphi_{\text{Krat}} : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ (Krattenthaler '01)

Example: $\pi = 67435281 \in \mathcal{S}_8(132)$



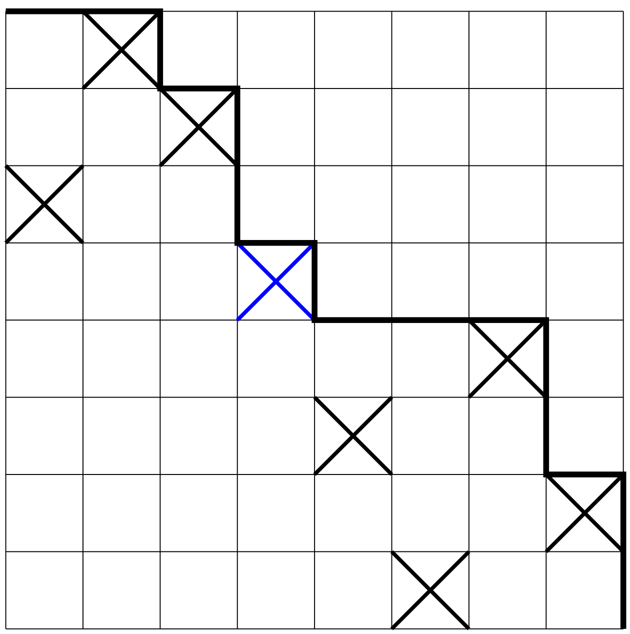
Fixed points <−−> centered tunnels

Excedances <−−> right tunnels



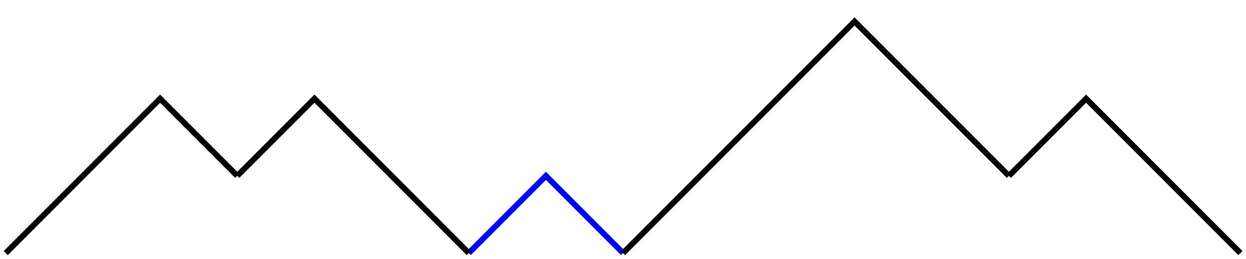
$\psi : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$

Example: $\pi = 23147586 \in \mathcal{S}_8(321)$



Fixed points <-->

hills (= peaks of height 1)

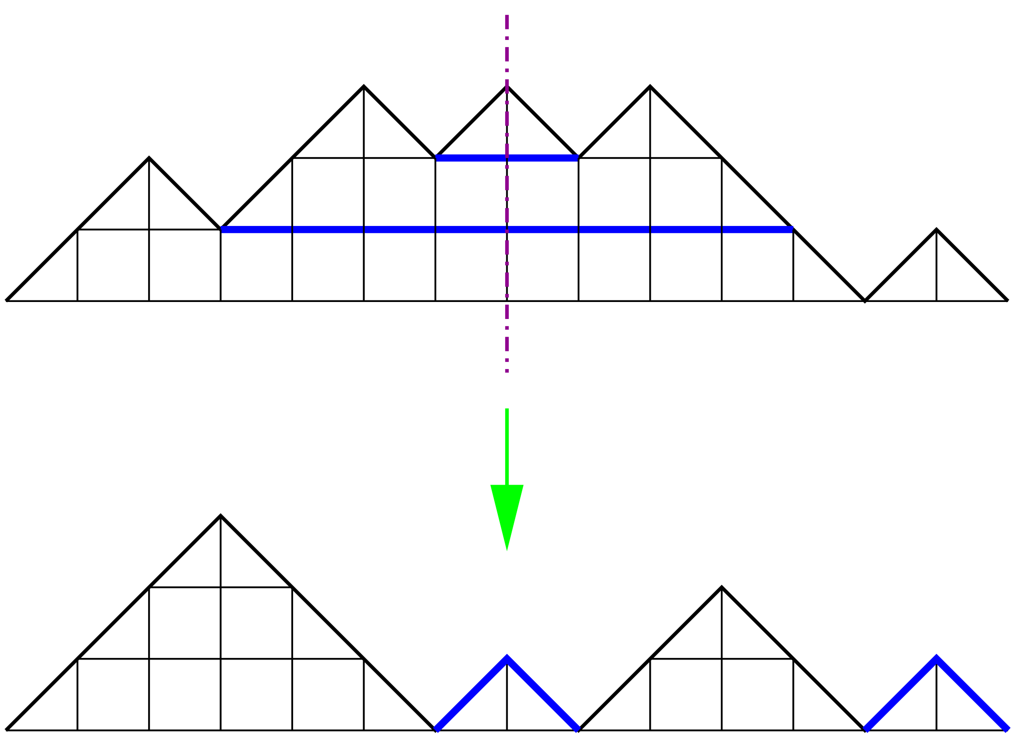


$\Phi_{\text{ED}} : \mathcal{D}_n \longrightarrow \mathcal{D}_n$ (Deutsch, E. '03)

1. Each u in D has a matching d (together they determine a tunnel).

2. Read the steps of D in zigzag: $1, 2n, 2, 2n-1, \ldots$

3. For each step, if its corresponding matching step has not yet been read, draw an up-step in $\Phi_{\text{ED}}(D)$. Otherwise, draw a down-step.



Φ_{ED} maps centered tunnels to hills.

□

More generally:

Deutsch, E. '03. Let

$$\alpha_r(\pi) := |\{i : \pi_i = i + r\}|$$

$$\beta_r(\pi) := |\{i : i > r, \pi_i = i\}|$$

Then, for any k, r, n,

$$|\{\pi \in \mathcal{S}_n(132) : \alpha_r(\pi) = k\}| = |\{\pi \in \mathcal{S}_n(321) : \beta_r(\pi) = k\}|$$

Proof uses a generalization of Φ_{ED}.

Another application of the bijection

Deutsch, E. '03.

$$1 + \sum_{n \geq 1} \sum_{\pi \in \mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} p^{\mathrm{des}(\pi)+1} z^n$$
$$= \frac{2(1 + xz(p-1))}{1 + (1 + q - 2x)z - qz^2(p-1)^2 + \sqrt{\spadesuit}}$$

where

$$\spadesuit = 1 - 2(1+q)z + [(1-q)^2 - 2q(p-1)(p+3)]z^2 - 2(1+q)(p-1)^2 z^3 + q^2(p-1)^4 z^4$$

Proof:

$\mathcal{S}_n(132)$	$\overset{\varphi_{\mathrm{Krat}}}{\longleftrightarrow}$	$\mathcal{D}_n$	$\overset{\Phi_{\mathrm{ED}}}{\longleftrightarrow}$	$\mathcal{D}_n$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	even up-steps
descents+1	$\leftrightarrow$	peaks	$\leftrightarrow$	"certain statistic"

("certain statistic": easier to enumerate)

Generalization to excedances

E. '02.

For any k, l, n,

$$|\{\pi \in \mathcal{S}_n(321) : \text{fp}(\pi) = k, \ \text{exc}(\pi) = l\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \text{fp}(\pi) = k, \ \text{exc}(\pi) = l\}|$$

$$\sum_{n\geq 1} \sum_{\pi \in \mathcal{S}_n(321)} x^{\text{fp}(\pi)} q^{\text{exc}(\pi)} z^n = \sum_{n\geq 1} \sum_{\pi \in \mathcal{S}_n(132)} x^{\text{fp}(\pi)} q^{\text{exc}(\pi)} z^n$$

$$= \frac{2}{1 + (1 + q - 2x)z + \sqrt{1 - 2(1 + q)z + (1 - q)^2 z^2}}$$

Original proof is analytical and uses nonstandard techniques in generating functions.

We will see a bijective proof.

A bijective proof (E., Pak '03) of

$$|\{\pi \in \mathcal{S}_n(321) : \text{fp}(\pi) = k, \text{exc}(\pi) = l\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \text{fp}(\pi) = k, \text{exc}(\pi) = l\}|$$

Composition of bijections:

$\mathcal{S}_n(132)$	$\xrightarrow{\varphi_{\text{Krat}}}$	$\mathcal{D}_n$	$\xleftarrow{\Psi_{\text{RSK}}}$	$S_n(321)$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	fixed points
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	excedances

$$\Psi_{\mathsf{RSK}} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$$

Step 1: RSK correspondence

$$\pi \mapsto (P, Q)$$

Example: $\pi = 23514687$

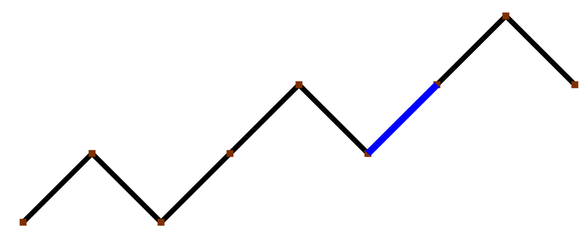
$$\begin{array}{|c|}\hline 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 8 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3} \end{array}$$

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3} \end{array} \qquad Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3} \end{array}$$

Step 2: $(P, Q) \mapsto \Psi_{\text{RSK}}(\pi)$

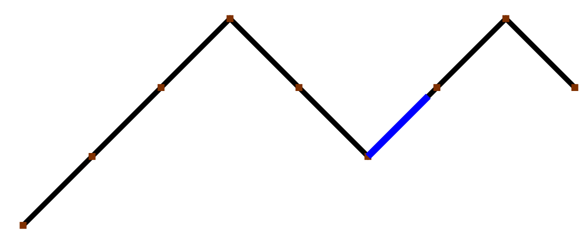
$P =$

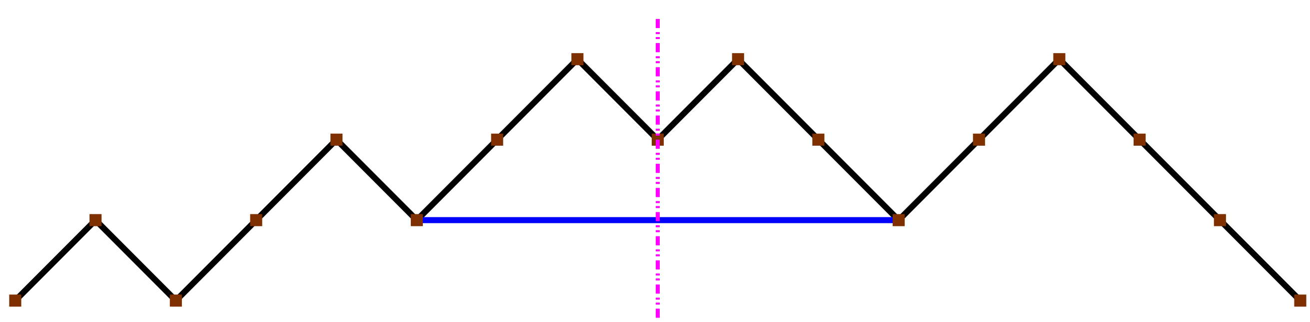
1	3	4	6	7
2	5	8		



$Q =$

1	2	3	6	7
4	5	8		

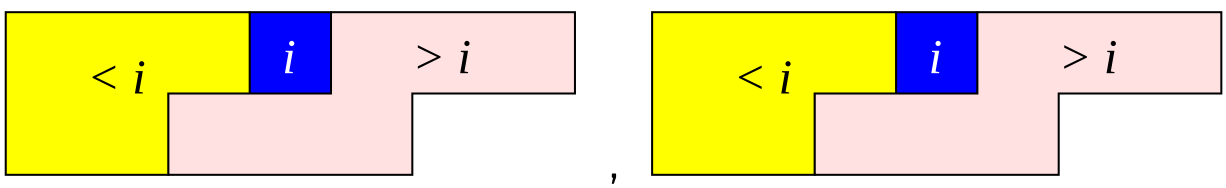




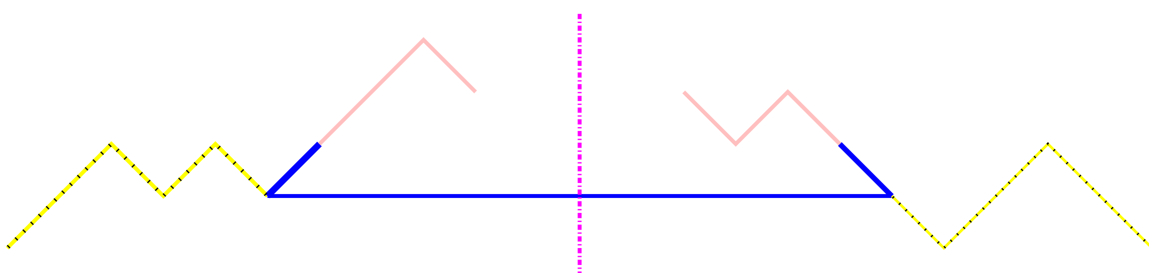
Ψ_{RSK} maps fixed points to centered tunnels:

$$\left.\begin{array}{c}\pi \text{ 321-avoiding} \\ \pi_i = i\end{array}\right\} \Rightarrow \pi = \underbrace{\cdots\cdots}_{<i} i \underbrace{\cdots\cdots\cdots}_{>i}$$

Then, both P and Q have shape



and i produces a centered tunnel.



It can also be checked that Ψ_{RSK} maps excedances to right tunnels.

$\square$

Statistics and simultaneous avoidance

$$F_{\sigma_1,\sigma_2} := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\sigma_1,\sigma_2)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} z^n$$

(define $F_{\sigma_1,\sigma_2,\sigma_3}$ similarly)

E. '03. Explicit expressions for F_{σ_1,σ_2} and $F_{\sigma_1,\sigma_2,\sigma_3}$, for any $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{S}_3$.

Examples:

$$F_{132,231} = \frac{1 - z - qz^2 + xqz^3}{(1 - xz)(1 - z - 2qz^2)}$$

$$F_{132,213} = \frac{1 - (1+q)z - 2qz^2 + 4q(1+q)z^3 - (xq^2 + xq + 5q^2)z^4 + 2xq^2z^5}{(1-z)(1-xz)(1-qz)(1-4qz^2)}$$

$$F_{123,132,213} = \frac{1+xz+(x^2-q)z^2+(-xq+q^2+q)z^3-x^2qz^4}{(1+qz^2)(1-3qz^2+q^2z^4)}$$

Idea of the proof: bijections between pattern-avoiding permutations and Dyck paths with certain restrictions, so that fp and exc correspond to statistics that are easier to enumerate.

Statistics and generalized patterns

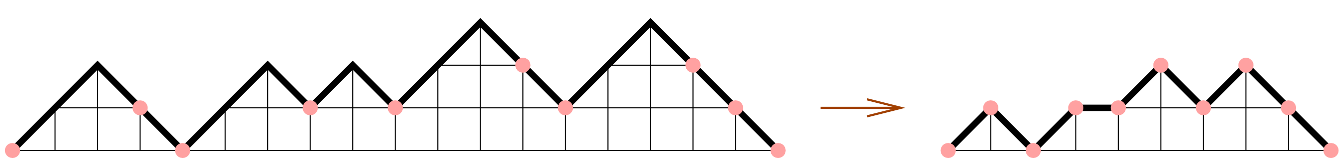
E., Mansour '03.
Bijection between $\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23)$ and $\mathcal{M}_n$.

$\widehat{\mathcal{D}}_n := \{D \in \mathcal{D}_n$ with no uuu (3 consec. up-steps)$\}$

$$\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23) \xrightarrow{\varphi_{\text{Krat}}} \widehat{\mathcal{D}}_n \xrightarrow{\theta_{\text{EM}}} \mathcal{M}_n$$

lds	$\leftrightarrow$	steps u and h
lis	$\leftrightarrow$	height+1
rises	$\leftrightarrow$	steps u

$$\theta_{\text{EM}} : \widehat{\mathcal{D}}_n \longrightarrow \mathcal{M}_n$$



$$\begin{aligned} uud &\rightarrow u \\ ud &\rightarrow h \\ d &\rightarrow d \end{aligned}$$

From the bijection,

E., Mansour '03.

$$\sum_{n\geq 0} \sum_{\substack{\pi\in\mathcal{S}_n(132,1\text{-}23,\\ 12\cdots(k+1))}} v^{\mathrm{lds}(\pi)} y^{\#\{\text{rises of } \pi\}} z^n$$

$$= \frac{U_{k-1}\left(\frac{1-vz}{2z\sqrt{vy}}\right)}{z\sqrt{vy}\,U_k\left(\frac{1-vz}{2z\sqrt{vy}}\right)}$$

where $U_m(\cos t) = \frac{\sin(m+1)t}{\sin t}$
(Chebyshev polynomials of the second kind).

Open questions

- Are there other statistics in restricted permutations having the same distribution for different patterns?

- Find distribution of statistics in permutations avoiding longer patterns (e.g., of length 4).

- Find $|\mathcal{S}_n(\sigma)|$ for other patterns (e.g. $\sigma =$ 1324).

- How nice is $|\mathcal{S}_n(\sigma)|$?

  Conj. (Noonan, Zeilberger '96): For all σ, $|\mathcal{S}_n(\sigma)|$ is a P-recursive function of n.

- In the proof of the Stanley-Wilf conjecture ($|\mathcal{S}_n(\sigma)| < \lambda^n$), the constant λ is very big.

  Conj. (Arratia): If $\sigma \in \mathcal{S}_k$, $|\mathcal{S}_n(\sigma)| < (k-1)^{2n}$.

- Find $L(\sigma) = \lim_{n\to\infty} \sqrt[n]{|\mathcal{S}_n(\sigma)|}$.
  Known:

  $L(\sigma) = 4$ if $\sigma \in \mathcal{S}_3$
  $L(12\cdots k) = (k-1)^2$
  $L(1342) = 8$
  Bóna: $L(12453) = (1+\sqrt{8})^2$