Number Theory Web Seminar

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Practical numbers

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Our story begins with **Fibonacci** in his book Liber Abaci in 1202. He noticed that for some numbers n, like 12, one could express each fraction m/n with $m \le n$ as a sum of distinct unit fractions with denominators dividing n:

$$\frac{1}{12} = \frac{1}{12}, \quad \frac{2}{12} = \frac{1}{6}, \quad \frac{3}{12} = \frac{1}{4}, \quad \frac{4}{12} = \frac{1}{3},$$

$$\frac{5}{12} = \frac{1}{6} + \frac{1}{4}, \quad \frac{6}{12} = \frac{1}{2}, \quad \frac{7}{12} = \frac{1}{4} + \frac{1}{3}, \quad \frac{8}{12} = \frac{1}{6} + \frac{1}{2},$$

$$\frac{9}{12} = \frac{1}{4} + \frac{1}{2}, \quad \frac{10}{12} = \frac{1}{3} + \frac{1}{2}, \quad \frac{11}{12} = \frac{1}{6} + \frac{1}{4} + \frac{1}{2}, \quad \frac{12}{12} = \frac{1}{1}.$$

This property is equivalent to each $m \le n$ being a sum of distinct divisors of n. In the example, just multiply each equation above by 12.

In 1948, **A. K. Srinivasan** called such numbers *practical*. He somewhat sarcastically claimed "The subdivisions of money, weight, and measures are often done with numbers such as 4, 12, 16, 20, and 28 which are usually thought to be so inconvenient as to deserve replacement with powers of 10." In his brief article he begins a study of the multiplicative nature of a practical number.

A multiplicative criterion was found by **W. Sierpiński** in 1955 and independently by **B. M. Stewart** in 1954:
Recursively define a sequence of numbers that contains 1 *and if it contains* m*, it also contains* mp *for primes* $p \le \sigma(m) + 1$*. This is precisely the sequence of practical numbers.*

Here σ is the usual sum-of-divisors function.

Fibonacci

A. K. Srinivasan

Wacław Sierpiński

From the criterion we see for example that every power of 2 is practical and that all practical numbers after 1 are even.

To prove the criterion, first note that if $p > \sigma(m) + 1$, then $p - 1$ cannot be written as a sum of divisors of mp, so the condition is necessary. To show sufficiency we prove a stronger result by induction:
If m is practical and $p \leq \sigma(m) + 1$ is a prime not dividing m, then every number up to $\sigma(mp^\alpha)$ can be expressed as a subset sum of the divisors of mp^α.

Assume every number up to $\sigma(m)$ is a subset sum of divisors of m, and let p be a prime not dividing m with $p \le \sigma(m) + 1$. Consider the number mp^α. The result holds for $\alpha = 0$, so assume it holds at α. Consider the interval

$$I_a = [ap^{\alpha+1}, ap^{\alpha+1} + \sigma(mp^\alpha)], \text{ where } a \le \sigma(m).$$

By the induction hypotheses, each number in this interval can be written as a subset sum of divisors of $mp^{\alpha+1}$. Further,

$$\sigma(mp^\alpha) = \sigma(m)\sigma(p^\alpha) \ge (p-1)\frac{p^{\alpha+1}-1}{p-1} = p^{\alpha+1} - 1.$$

Thus, the intervals I_a can be glued together and we can represent all numbers up to $\sigma(m)p^{\alpha+1} + \sigma(mp^\alpha) = \sigma(mp^{\alpha+1})$.

So is there a *practical number theorem*, which gives an asymptotic for $N(x)$, the number of practical numbers in $[1,x]$?

Srinivasan computed that $N(200) = 50$ and was not sure if $N(x) = o(x)$. In 1950, **Erdős** claimed that $N(x) = o(x)$. Here is likely what he was thinking:

A number n normally has close to $\log\log n$ prime factors, whether counted with or without multiplicity. This is a famous theorem of **Hardy** and **Ramanujan**. So, for example, the integers n with more than $1.1\log\log n$ prime factors have density 0. But if n has at most $1.1\log\log n$ prime factors, then the number of divisors of n is $\le 2^{1.1\log\log n}$ and the number of subset sums of divisors is $\le 2^{2^{1.1\log\log n}} < n$ (since $1.1\log 2 < 1$). So, there are fewer than n subset sums of the divisors of n, so n cannot be practical.

Srinivasa Ramanujan

G. H. Hardy

Paul Erdős

Looking at a more quantitative version of the **Hardy–Ramanujan** theorem would give the bound

$$N(x) \le \frac{x}{(\log x)^{\eta+o(1)}}, \qquad \eta = 1 - \frac{1+\log\log 2}{\log 2} = 0.08607\cdots.$$

However, the "correct" exponent on $\log x$ is essentially 1, as shown by **Geráld Tenenbaum** (1986, 1995):

$$N(x) = \frac{x}{\log x}(\log\log x)^{O(1)}.$$

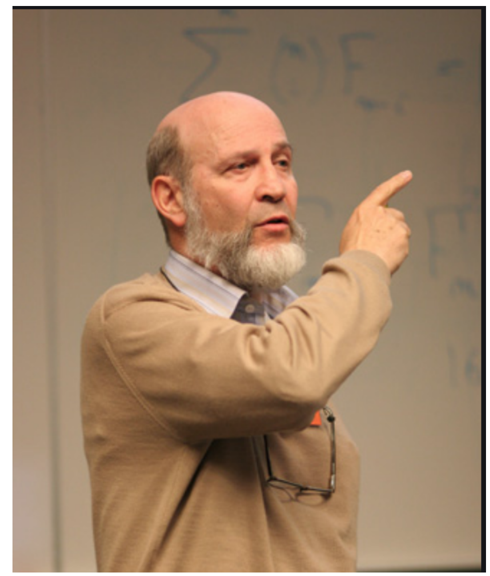
In 1997 **Éric Saias** got rid of the loglog factors:

$$N(x) \asymp \frac{x}{\log x}.$$

Meanwhile, in 1991, **Maurice Margenstern** had conjectured that there is a constant $c > 0$ with

$$N(x) \sim c \frac{x}{\log x}, \quad x \to \infty,$$

and on the basis of computing to 10^{13} that $c \approx 1.341$.



In 2015, **Andreas Weingartner** showed that this indeed holds for some c, and in 2020, he showed that $c = 1.33607\cdots$.

In fact, we have

$$c = \frac{1}{1 - e^{-\gamma}} \sum_{n \text{ practical}} \frac{1}{n} \left(\sum_{p \le \sigma(n)+1} \frac{\log p}{p-1} - \log n \right) \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right).$$

In her 2012 dissertation, **Thompson** considered those n where the polynomial $t^n - 1$ has divisors in $\mathbb{Z}[t]$ of each degree $m \le n$. For example, $t^{12} - 1$ has the cyclotomic polynomials $\Phi_d(t)$ as its irreducible divisors for $d \mid 12$. They have degrees 1, 1, 2, 2, 2, and 4, and subsequence sums of these give all $m \le 12$.

Lola Thompson

Thus, she said a number n is φ-*practical* if for each $m \le n$ there is a subset S of the divisors of n such that $m = \sum_{d \in S} \varphi(d)$. These numbers are somewhat trickier than the practical numbers, as there is no direct analogue of the **Sierpiński–Stewart** multiplicative criterion. There, if you remove the top prime from a practical number, the quotient is still practical. With φ-practical numbers note that $315 = 3^2 \times 5 \times 7$ is one, but $45 = 3^2 \times 5$ is not. Also, increasing the exponent on a prime keeps a practical number practical, but not so with φ-practicals: 3 is φ-practical, but 9 is not.

Nevertheless, **Thompson** was able to use the **Saias** machinery to show that

$$N_\varphi(x) \asymp \frac{x}{\log x},$$

where $N_\varphi(x)$ is the number of φ-practical numbers in $[1,x]$. She conjectured the asymptotic $N_\varphi(x) \sim cx/\log x$ for some $c > 0$. Numerically it seemed that $c \approx 1$.

In 2016, **P, Thompson, & Weingartner** proved the conjecture. Though we didn't compute c, we gave a heuristic that $c \approx 0.96$.

Another application of the ideas behind practical numbers is to numbers with "dense" divisors. These are numbers n whose increasing sequence of divisors $1 = d_1 < d_2 < \cdots < d_t = n$ satisfy $d_i/d_{i-1} \le 2$ for $i = 2, \ldots, t$. (This condition is stronger than in the **Erdős** "propinquity" problem, where he conjectured that asymptotically all integers have two divisors d, d' with $1 < d'/d \le 2$; this was proved by **Maier & Tenenbaum** in 1984.)

Helmut Maier

The integers with dense divisors are sparser than the practicals, in fact one can show that they are a proper subset. Similar to the **Sierpiński–Stewart** criterion, the integers with dense divisors are recursively built from the rule that 1 is in the set, and if m is in, so is mp for all primes $p \le 2m$. That is, we replace "$\sigma(m) + 1$" in the rule for practical numbers with "$2m$".

After many papers by **Tenenbaum**, **Saias**, and **Weingartner**, we now know the asymptotic distribution of the integers with dense divisors, it is of the shape $c'x/\log x$. This work played a prominent role in the work on practical numbers, and φ-practicals.

Because of the **Margenstern** conjecture (and **Weingartner** theorem) that $N(x) \sim cx/\log x$, it seems natural to ask similar questions for practicals as we ask for primes.

For example, do we have infinitely many twin practicals, namely n and $n+2$ are both practical? **Margenstern** (1991) proved that this is in fact the case, and **P & Weingartner** (2020) showed that the number $N_2(x)$ of such n in $[1, x]$ satisfies

$$\frac{x}{(\log x)^{9.5367}} \ll N_2(x) \ll \frac{x}{(\log x)^2}.$$

The same holds for any even gap $h > 0$, with implied constants depending on h. Presumably $N_2(x) \sim c_2 x/(\log x)^2$.

To show that the number of twin practicals to x is $\ll x/(\log x)^2$, we write a pair of twin practicals $n, n+2$ with $n \le x$ as $mq, m'q'$, where $m, m' \in [x^{1/7}, x^{1/3}]$, with the smallest prime factor of q at least the largest prime factor of m, and similarly for m', q'. The proof is largely completed by a sieve argument: for a given choice of m, m', count values of $q \le x/m$ with q having only large prime factors, $qm + 2 \equiv 0 \pmod{m'}$, and the prime factors of $(qm+2)/m'$ are all large. There are some technical hurdles in eliminating smaller order factors in the expression $x/(\log x)^2$.

Here's how we show there are many twin practicals. We can show there are $\gg x/(\log x)^2$ pairs m_1, m_2 of practical numbers in $(\sqrt{x}/2, \sqrt{x}]$ with $\gcd(m_1, m_2) = 2$. Given such a pair, there are $a_1, a_2 \leq \sqrt{x}$ with $a_1 m_1 - a_2 m_2 = 2$. But multiplying a practical number m by a number $a < 2m$ has am also practical. Thus, we have the twin practicals $a_1 m_1, a_2 m_2$.

It would seem that we have created $\gg x/(\log x)^2$ twin practicals below x, but one needs to wonder if the same twin practical pair is counted multiple times. Since there are many practical numbers in $(\sqrt{x}/2, \sqrt{x}]$, we can discard those with many prime factors, so $\Omega(m_1), \Omega(m_2)$ are under control. Also, for m_1 fixed, the map that sends m_2 to a_1 is at most 2-to-1, and similarly for a_2. Thus, we see many different numbers a_1, a_2, and again we can discard those with $\Omega(a_1), \Omega(a_2)$ large. Thus, the number of representations for a given pair of twin practicals is at most "poly-log" in size, which leads to our theorem.

Guo & Weingartner (2018) showed that the number $N_\pi(x)$ of primes $p \le x$ with $p-1$ practical satisfies

$$\frac{\pi(x)}{(\log x)^{4.7684}} \ll N_\pi(x) \ll \frac{\pi(x)}{(\log x)^{0.08607}}.$$

This has been improved (**P & Weingartner** (2020)) to

$$\frac{\pi(x)}{(\log x)^{2.1648}} \ll N_\pi(x) \ll \frac{\pi(x)}{\log x}.$$

Presumably the new upper bound is tight.

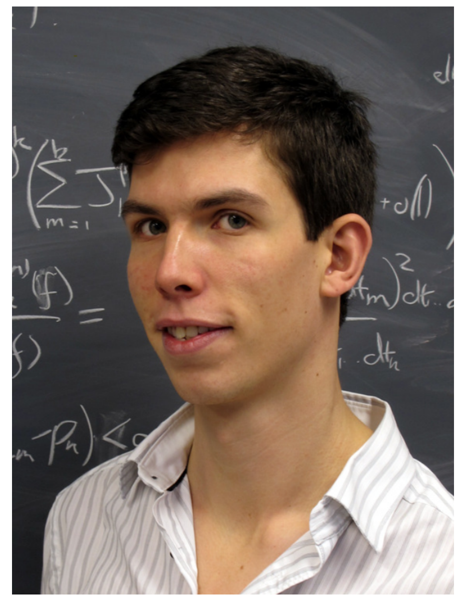
For the upper bound proof, write a practical number n as mq where q is the largest prime factor of n. Then if $n+1=p$ is prime, we are asking, for a given m, for primes q with $mq+1$ prime. This almost does it, but there are some technicalities to deal with the cases when q is small (so m is smooth) or when $m/\varphi(m)$ large.

Here is how we attacked the lower bound. If n is a practical number and p is a prime with $p < n^2$ and $p \equiv 1 \pmod{n}$, then $p - 1 = an$ is practical. Unfortunately we don't know that there are any primes at all satisfying this, though it is conjectured.

However, if we let n be a little below $\sqrt{x}$ and average over many practical numbers n in this range, the **Bombieri–Vinogradov** theorem comes to the rescue and shows that there are many such p, n pairs, with $p \leq x$.

The trouble with this argument is that it is not guaranteed that $p - 1 = an$ is practical, since now a can be a fair bit above n. However, most values of a would work, only those with a very large prime factor are ruled out, and we can use sieve methods to prove this.

Using a brand new theorem of **Maynard** instead of the **Bombieri–Vinogradov** theorem would eliminate the need to worry about the numbers a that appear, they can be assumed to be $< n$.



Enrico Bombieri **Askold Ivanovich Vinogradov** **James Maynard**

What about an analogue of the **Goldbach** conjecture for practical numbers?

Margenstern (1991) conjectured that every even number is the sum of two practical numbers, and that every odd number > 1 is the sum of a prime and a practical number.

Melfi (1996) proved the even part of the conjecture.

Giuseppe Melfi

And now **P & Weingartner** (2020) showed that the odd part holds for sufficiently large numbers. Here is how we did it.

Let a be an odd number in the interval $(x, 2x]$. Let n run over practical numbers somewhat, but not much smaller than $x^{1/2}$. By the **Bombieri–Vinogradov** theorem there are many pairs n, p where $p \le x$ and $p \equiv a \pmod{n}$. Then, since $a > x \ge p$, we have $a - p > 0$ and divisible by the practical number n; say $a - p = bn$. Then, unless b is divisible by a large prime, the number bn is practical and we have succeeded in representing a as the sum of a prime and a practical. And we can use sieve methods to show that most of the time b is not divisible by a large prime.

So, one might wonder if the full **Margenstern** conjecture could be proved. That is, there are no exceptions, *every* odd number > 1 is the sum of a prime and a practical.

This would be difficult using our proof since the **Bombieri–Vinogradov** theorem depends on **Siegel**'s theorem, and so is ineffective. There are ways around this perhaps, but maybe it is more promising to assume the Extended Riemann Hypothesis (**ERH**). There we have the very nice prime number theorem for residue classes:

$$\left|\pi(x; n, a) - \frac{1}{\varphi(n)}\mathrm{li}(x)\right| \leq \sqrt{x}\log(n^2 x), \quad \gcd(n, a) = 1.$$

A back-of-the-envelope calculation shows that assuming the **ERH**, the **Margenstern** conjecture holds for all odd numbers $> e^{10{,}000}$.

Tomás Oliveira e Silva told us he checked up to 10^9, and there are no exceptions. Well, it's a start!

Actually, we found a cool method for numerically checking up to much higher levels, though $e^{10,000}$ is definitely out of reach. Here's what we do: Take a power of 2, say 2^k and compute the largest of the least primes for each $a \pmod{2^k}$ as a varies over odd numbers $< 2^k$. As a toy example, say $k = 4$. Then the largest of the least primes (mod 16) is 41 in the residue class 9. (The only residue classes to look at really are 1, 9, and 15.)

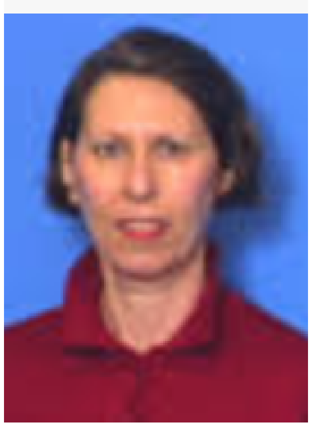
So say p is the largest of these least primes. **Linnik**'s theorem gives an upper bound for p, but numerically in practice, it will not be very much larger than 2^k, say $< k^2 2^k$. Once p is found, we have done all of the work to verify the conjecture for odd numbers $a \in (p, 2^{2k+1})$. Indeed, if a is odd in this range, there is a prime $q \equiv a \pmod{2^k}$ with $q \le p$, and so $a = q + 2^k b$, with $b < 2^{k+1}$. So, $2^k b$ is practical. (With our toy example, odds in the interval $(41, 512)$ are representable.)

To find these primes, we check which residue classes (mod 2^k) are filled by primes below $k^2 2^k$, if we find all odd classes, we're done. (Actually, if we go merely to $3k2^k$, there should be $< 0.02\%$ of unrepresented classes, and we can then search separately over these.)

Using this, we have checked the conjecture up to 2^{53} and could go further.

A famous conjecture of **Landau** is that between consecutive squares there is always at least one prime. Could such a theorem be proved for the practical numbers?

Yes, and this was done in 1984 by **Hausman & Shapiro**, and a small improvement was found by **Melfi** in 1995. It would be nice to prove that if $\epsilon > 0$ is fixed and x is sufficiently large depending on ϵ, then there is a practical number in the interval $(x, x + x^{\epsilon})$.

Miriam Hausman   **Harold N. Shapiro**

This would follow from the following possible argument. Let 2^k be the first power of 2 above $x^{\epsilon/2}$ and consider the numbers $b2^k$ where $b \in (x/2^k, (x+x^\epsilon)/2^k)$. If just one of these integers b is $x^{\epsilon/2}$-smooth, then $b2^k$ is practical and between x and $x+x^\epsilon$.

There is a featured conjecture in **Granville**'s survey on smooth numbers: *Given numbers* α, β *in* $(0,1)$*, if* x *is sufficiently large depending on* α, β*, there is an* x^α*-smooth number in the interval* $(x, x+x^\beta)$*.*

So, this conjecture would imply that short intervals contain practicals. Maybe there's an unconditional proof?

In conclusion:

Some people have used the forced idleness of the pandemic to do great things, as witnessed by this stimulating Number Theory Web Seminar.

One might say at least that I have been doing practical things.

Thank You