Dartmouth College  
Mathematics 101  
Homework 1 (due Wednesday, Sept 29)

1. Let $G$ be a group and $H$, $K$ subgroups of $G$. The \textit{normalizer} of $H$ in $G$, denoted in Lang by $N_H$ (with $G$ implicit), is defined by $N_H = \{ g \in G \mid gHg^{-1} = H \}$. In other books you might see this written as $N_G(H)$.

Show that if $K \subseteq N_H$, then $HK = KH$ is a subgroup of $G$, and $H \triangleleft KH$. Of course in particular, if $H \triangleleft G$, then $HK$ is a subgroup of $G$.

2. Let $G$ be a group, and $H_1, H_2, \ldots, H_n$ subgroups of $G$. Suppose that

(a) $G = H_1H_2\cdots H_n$

(b) $H_{i+1} \cap (H_1H_2\cdots H_i) = \{e\}$ for all $1 \leq i \leq n - 1$.

(c) The elements of $H_i$ commute with those of $H_j$ for all $i \neq j$.

Show that the map $\varphi : H_1 \times \cdots \times H_n \rightarrow G$ given by $\varphi((h_1, \ldots, h_n)) = h_1h_2\cdots h_n$ is an isomorphism.

3. In the previous problem show that condition (c) can be replaced with the following:

(c’) $H_i \triangleleft G$ for $1 \leq i \leq n$

Note: it is not the case that (c) and (c’) are equivalent, however, you can show that (a), (b), (c) are equivalent to (a), (b), (c’).

As we remarked in class, condition (b) is often replaced by the symmetric, but more stringent condition that $H_i \cap (\cup_{i \neq j} H_j) = \{e\}$.