1. Let \( A = \mathbb{Z} \) and \( p = p\mathbb{Z} \) with \( p \) a prime in \( \mathbb{Z} \). We have characterized the localization \( A_p = \mathbb{Z}_p \) as \( \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z}, \ p \nmid b, \ \gcd(a, b) = 1\} \).

(a) Show that every nonzero element in \( \mathbb{Z}_p \) can be written uniquely as \( p^{\nu}u \) where \( \nu \) is a nonnegative integer and \( u \in \mathbb{Z}_p^\times \). You may of course assume unique factorization in \( \mathbb{Z} \).

(b) Characterize all the ideals of \( \mathbb{Z}_p \), and confirm that \( \mathbb{Z}_p \) has a unique maximal ideal.

(c) Show that \( \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} \).

2. Let \( A \) be a commutative ring with identity.

(a) Suppose that for each prime ideal \( \mathfrak{p} \) in \( A \), the local ring \( A_{\mathfrak{p}} \) has no nonzero nilpotent elements. Show that \( A \) has no nonzero nilpotent elements. Hint: Show that for an element \( x \in A \), the set \( \text{Ann}(x) = \{y \in A \mid yx = 0\} \) is an ideal of \( A \). Ann\((x)\) is called the annihilator of the element \( x \).

(b) Proof or counterexample: If for each prime \( \mathfrak{p} \) of \( A \), each localization \( A_{\mathfrak{p}} \) is an integral domain, then \( A \) is an integral domain.

3. Let \( A \) be an integral domain, \( S \subseteq A \) a multiplicative subset containing 1 (but not containing 0).

(a) Show that \( S^{-1}A \) is an integral domain.

(b) Show that if \( A \) is a PID (every ideal is principal), so is \( S^{-1}A \).

4. Consider the localization of \( \mathbb{Z}[x] \) at the prime ideal \((x)\).

(a) Describe the elements of \( \mathbb{Z}[x]_{(x)} \).

(b) Is \((x)\) maximal in \( \mathbb{Z}[x]_{(x)} \)? If so, describe the resulting quotient field.

(c) How does \( \mathbb{Z}[x]_{(x)} \) compare to \( \mathbb{Q}[x]_{(x)} \)?

5. Let \( A \) be a commutative ring with identity, and let \( X \) be the set of all prime ideals in \( A \). \( X \) is called the prime spectrum of \( A \), written \( \text{Spec}(A) \). For each subset \( E \subseteq A \), let \( V(E) \) denote the set of primes ideals of \( A \) which contain \( E \). The properties below
demonstrate that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. This topology is called the Zariski topology on $Spec(A)$.

Prove that:

(a) If $I = \langle E \rangle$ is the ideal generated by $E$, then $V(I) = V(E)$.

(b) Show that $V(0) = X$ and $V(1) = \emptyset$.

(c) If $\{E_i\}_{i \in I}$ is any family of subsets of $A$, then $V(\bigcup_i E_i) = \bigcap_{i \in I} V(E_i)$.

(d) For any ideals $I, J$ of $A$, show that $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.