Homework problems, due May 4, 2009

1. Let \( \varphi \) denote Euler’s function. Show that for each positive integer \( m \), we have that the set \( \{ n : m \mid \varphi(n) \} \) has asymptotic density 1.

2. Let \( \sigma \) denote the sum-of-divisors function, so that \( \sigma(n) = \sum_{d|n} d \). Do the above problem with \( \sigma \) instead of \( \varphi \).

3. Let \( s(n) = \sigma(n) - n \), the sum-of-proper-divisors function. Show that for each positive integer \( m \), the set \( \{ n : m \mid n, \ m \nmid s(n) \} \) has asymptotic density 0.

4. Let \( \lambda(n) \) denote the universal exponent of the group \( (\mathbb{Z}/n\mathbb{Z})^\times \). That is, \( \lambda(n) \) is the least positive integer such that \( a^{\lambda(n)} \equiv 1 \pmod{n} \) for all integers \( a \) coprime to \( n \). (It can also be defined as the order of the largest cyclic subgroup of \( (\mathbb{Z}/n\mathbb{Z})^\times \).) There is a formula for \( \lambda(n) \) that is similar to the one for \( \varphi(n) \). In particular, for a prime power \( p^a \), we have \( \lambda(p^a) = \varphi(p^a) \) unless \( p = 2, a \geq 3 \), in which case, \( \lambda(2^a) = 2^{a-2} \). And if \( n \) has the prime factorization \( p_1^{a_1} \cdots p_k^{a_k} \) with distinct primes \( p_1, \ldots, p_k \), then

\[
\lambda(n) = \text{lcm}\{\lambda(p_1^{a_1}), \ldots, \lambda(p_k^{a_k})\}.
\]

For example, \( \lambda(1000) = \text{lcm}\{2, 5^2 \cdot 4\} = 100 \), and \( \lambda(1001) = \text{lcm}\{6, 10, 12\} = 60 \). Show that the set of odd numbers \( n \) with

\[
2^{\lambda(n)/2} \not\equiv 1 \pmod{n}
\]

has asymptotic density 0. (Hint: If \( p \equiv 1 \pmod{8} \), then 2 is a quadratic residue for \( p \).)

5. Let

\[
A = \{ n : n \mid 2^k - 1 \text{ for some positive integer } k \},
\]

\[
B = \{ n : n \mid 2^k + 1 \text{ for some positive integer } k \}.
\]

Show that \( A \) has asymptotic density 1/2 and \( B \) has asymptotic density 0. (Hint: If \( p \equiv 7 \pmod{8} \), then 2 is a quadratic residue for \( p \).)