Math 113, Spring 2002,
Take-home Final Exam
due on Monday, June 3, 2002, 4:00 PM

Instructions: You may use the books put on reserve for the class at the library. The written form that you hand in should be well articulated and coherent, and should reflect your understanding of the assigned problems. The only person you may discuss the exam with is your instructor. A violation of this will be treated as a violation of the Honor Principle. Write on one side of your paper, and make sure that your name is on every page.

1. Let $I$ be any set and let $l^2(I)$ to be the set of all functions $x: I \to \mathbb{C}$ such that $x(i)=0$ for all but a countable number of $i$'s and $\sum_{i \in I} |x(i)|^2 < \infty$. Define:

$$\langle x, y \rangle = \sum_{i \in I} x(i) \overline{y(i)}, \quad \text{for } x, y \in l^2(I).$$

Show that $l^2(I)$ is a Hilbert space.

2. Consider two complex Hilbert spaces $H$ and $K$, and define $\mathcal{L}(H,K)$ to be the set of all maps (just maps!) $T : H \to K$ such that there exists a map (just map!) $T^* : K \to H$ such that

$$\langle Th, k \rangle = \langle h, T^* k \rangle, \quad \text{for all } h \in H, k \in K.$$

(a) Show that $T$ is actually a linear operator.

(b) For each $h \in H$, $\|h\| = 1$, consider $f_h : K \to \mathbb{C}$, $f_h(k) = \langle k, Th \rangle$. Verify that the family $\{f_h | h \in H, \|h\| = 1\}$ satisfies the hypothesis of PUB and apply it to deduce that $T$ is also a bounded linear operator.

3. Let $l^\infty = l^\infty(\mathbb{N})$. Define a linear operator $T : l^\infty \to l^\infty$ by

$$T((x_n)_n) = (y_n)_n, \quad \text{where } y_n = \frac{x_n + x_{n+1}}{n}, \quad n = 1, 2, \ldots.$$

Find the spectrum of $T$. (Hint. Consider a basis $\{e_n\}_n$ in $l^\infty$ and write $T$ in ‘matrix’ form.)
4. (Continuous functional calculus) Recall that a \(*\)-morphism \(\varphi : \mathcal{A} \to \mathcal{B}\) between two \(C^*\)-algebras \(\mathcal{A}\) and \(\mathcal{B}\) is a linear map that preserves the multiplication \((\varphi(aa') = \varphi(a)\varphi(a'))\) and the involution \((\varphi(a^*) = \varphi(a)^*)\). Consider a self-adjoint element \(a\) of a \(C^*\)-algebra \(\mathcal{A}\). Show that there is a unique isometric \(*\)-homomorphism 
\[
\Phi : C(\sigma(a)) \longrightarrow \mathcal{A}, \ f \mapsto f(a) =: \Phi(f),
\]
such that for any polynomial \(p(z) = \sum_{i=0}^{n} \beta_i z^i\) one has \(\Phi(p) = p(a) = \sum_{i=0}^{n} \beta_i a^i\).

(Hint. Recall that \(\sigma(a) \subseteq [0, \infty)\). Prove that \(\Phi\) is an isometric \(*\)-homomorphism when restricted to polynomials. Use Stone-Weierstrass to finish the argument.)

5. Let \(H\) be a Hilbert space with inner product \(\langle , \rangle\).

(a) Show the polarization identity:
\[
4\langle Tx, y \rangle = \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle + i\langle T(x+iy), x+iy \rangle - i\langle T(x-iy), x-iy \rangle,
\]
for all \(T \in \mathcal{B}(H)\), and all \(x, y \in H\).

(b) Consider \(T, S \in \mathcal{B}(H)\) such that \(\langle Tx, x \rangle = \langle Sx, x \rangle\), for all \(x \in X\). Use (a) to show that \(\langle Tx, y \rangle = \langle Sx, y \rangle\), for all \(x, y \in X\). Conclude that \(T = S\).

(c) If \(\langle Tx, x \rangle \geq 0\), for all \(x \in X\), show that \(T = T^*\).

(d) Use the above and the continuous functional calculus (for \(f(t) = \sqrt{t}\)) to prove the following:

Recall that an element \(a\) of a \(C^*\)-algebra \(\mathcal{A}\) is called positive if there exist \(b, c \in \mathcal{A}\) such that \(a = b^* b = c^2\). Consider \(\mathcal{A} = \mathcal{B}(H)\), the \(C^*\)-algebra of bounded linear operators on a Hilbert space \(H\), and \(T \in \mathcal{B}(H)\). Show that \(T\) is positive if and only if \(\langle Th, h \rangle \geq 0\), for all \(h \in H\).