Optional Assignment on Nets
Please do NOT turn in.

1. Suppose that $X$ is a first countable space. Show that each $x \in X$ has a neighborhood basis of open sets $\{ U_n \}_{n=1}^{\infty}$ such that $U_{n+1} \subseteq U_n$.

2. Let $X$ and $Y$ be 1st countable spaces.

(a) Show that $\mathcal{O} \subseteq X$ is open in $X$ if and only if every sequence converging to some $x \in \mathcal{O}$ is eventually in $\mathcal{O}$.

(b) Show that $F \subseteq X$ is closed if and only if every convergent sequence in $F$ converges to a point in $F$.

(c) Show that $f : X \rightarrow Y$ is continuous if and only if whenever $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$ then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in Y$.

(d)* Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $X$. Show that if $\{x_n\}_{n=1}^{\infty}$ has a convergent subnet, then $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence. (Hint: if $\{x_n\}_{n=1}^{\infty}$ has an accumulation point, then it must have a convergent subsequence.)

(e)* Show that 1st countability is required in part (d). (Hint: let $\ell^\infty$ denote the set of bounded sequences. If $\alpha = \{\alpha_n\}_{n=1}^{\infty} \in \ell^\infty$, let $I_\alpha$ be any closed bounded interval in $\mathbb{R}$ such that $\alpha_n \in I_\alpha$ for all $n$. Set

$$Z = \prod_{\alpha \in \ell^\infty} I_\alpha.$$ 

Consider the sequence $\{x_n\}_{n=1}^{\infty}$ in the compact space $Z$ defined by $x_n(\alpha) = \alpha_n$. )