Homework Assignment #6  
Due Wednesday, March 10th.

1. Let \( \{ e_j \}_{j \in J} \) be an orthonormal basis for \( \mathcal{H} \). Show that \( U \) is unitary if and only if \( \{ U e_j \}_{j \in J} \) is an orthonormal basis for \( \mathcal{H} \).

2. Suppose that \( P \) and \( Q \) are projections in \( B(H) \). We say that \( P \perp Q \) if \( P(H) \perp Q(H) \) and that \( P \leq Q \) if \( P(H) \subset Q(H) \).

   (a) Show that the following are equivalent.
      
      (i) \( P \perp Q \).
      
      (ii) \( PQ = QP = 0 \).
      
      (iii) \( P + Q \) is a projection.

   (b) Show that the following are equivalent.

      (i) \( P \leq Q \).

      (ii) \( PQ = QP = P \).

      (iii) \( Q - P \) is a projection.

   (Hint: Note that \( PQP \) is a positive operator. Also \( PQP = PQ(PQ)^* \) so that \( PQP = 0 \) if and only if \( PQ = QP = 0 \).)

3. Work E 3.3.1 in the text.

4. Work E 3.3.2 in the text.

5. Work E 3.3.4 in the text.