1. For fixed $k$, give the exponential generating function for the number of surjective maps from $[n]$ onto $[k]$.

2. Let $f(n)$ be the number of words of length $n$ on the alphabet $\{a, b, c, d\}$ that contain $a$ an odd number of times. Find an expression for $F(x) = \sum_{n \geq 0} f(n) x^n$ and also a formula for $f(n)$.

3. Given two sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$, their Hadamard product is the sequence $\{a_n b_n\}_{n \geq 0}$. Show that if $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ have rational generating functions, then so does their Hadamard product.

4. A set partition of $[n]$ is called noncrossing if it contains no two blocks $B$ and $B'$ such that $i, k \in B$ and $j, l \in B'$ for some $i < j < k < l$. Show that the number of noncrossing partitions of $[n]$ equals the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

5. (a) Find the bivariate generating functions $A(u, z) = \sum_{D \in \mathcal{D}} u^{\ell(D)} z^{||D||}$ and $B(u, z) = \sum_{D \in \mathcal{D}} u^{r(D)} z^{||D||}$, where $\mathcal{D}$ is the class of Dyck paths, and for $D \in \mathcal{D}$, $||D||$ is half of the number of steps, $\ell(D)$ is the number of up-steps before the first down-step, and $r(D)$ is the number of times that $D$ returns to the $x$-axis (the starting point does not count as a return).

(b) Give a bijection that explains why $A(u, z) = B(u, z)$.

6. Show that $e^x = \sum_{n \geq 0} \frac{x^n}{n!} \in \mathbb{C}[[x]]$ is not algebraic.

7. Consider two crossing lines in the plane with slopes 1 and $-1$, forming an X-shape. Place $n$ points anywhere on these lines, with no two of them having the same $x$- or $y$-coordinate, and label them 1, 2, \ldots, $n$ by increasing $y$-coordinate. Reading the labels of the points by increasing $x$-coordinate determines a permutation. For example, in the picture below we get 2 1 2 10 4 9 6 8 7 5 11 13 3 1.

Let $r_n$ be the number of permutations of $[n]$ that can be obtained in this way (note that $r_3 = 6$ and $r_4 = 20$). Find an ordinary generating function or an expression for $r_n$. 