Conservative Vector Fields

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Conservative Vector Field

\( \mathbf{F} \) is a **conservative vector field** if there is a scalar function \( f \) such that

\[
\mathbf{F} = \nabla f
\]

The function \( f \) is called a **potential function** for the vector field.
Definitions

- A path $C$ is **simple** if it doesn't cross itself.

- A region $D$ is **open** if it doesn't contain any of its boundary points.

- A region $D$ is **connected** if we can connect any two points in the region with a path that lies completely in $D$. 
Path-Independent Line Integrals

A continuous vector field \( \mathbf{F} \) has \textbf{path-independent line integrals} if

\[
\int_{C_1} \mathbf{F} \cdot ds = \int_{C_2} \mathbf{F} \cdot ds
\]

for any two simple, piecewise \( C^1 \), oriented curves in the domain of \( \mathbf{F} \) with the same endpoints.
Path-Independent Property

**Theorem:** Let $\mathbf{F}$ be a continuous vector field. Then $\mathbf{F}$ has a path-independent line integral if and only if

$$\oint_C \mathbf{F} \cdot ds = 0$$

for every piecewise $C^1$, simple, closed curves $C$ in the domain of $\mathbf{F}$. 
Fundamental Theorem of Line Integrals

Suppose that \( C \) be a \( C^1 \) oriented path given by \( c(t), \ a \leq t \leq b \). And suppose that \( f \) is a function whose gradient vector, \( \nabla f \), is continuous on \( C \). Then,

\[
\int_C \nabla f \cdot ds = f(c(b)) - f(c(a))
\]

Note that \( c(a) \) represents the initial point on \( C \) while \( c(b) \) represents the final point on \( C \).
Simply-Connected

A region $R$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is **simply-connected** if it consists of a single connected piece and if every simple closed curve $C$ in $R$ can be continuously shrunk to a point while remaining in $R$ throughout the deformation.
Test for a vector field to be conservative

Let $\mathbf{F}$ be a vector field of class $C^1$ whose domain is simply-connected region $R$ in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F} = \nabla f$ for some scalar-valued function $f$ of class $C^2$ on $R$ if and only if

$$\nabla \times \mathbf{F} = 0$$

for all points of $R$. 
Path-Independence and Conservative fields

If $\mathbf{F}$ is a continuous vector field on an open connected region $D$ and if $\int_C \mathbf{F} \cdot ds$ is independent of path (for any path in $D$) then $\mathbf{F}$ is a conservative vector field on $D$. 