Matrices and Coordinates

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Matrix Multiplication

If $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix then the product $AB$ is the matrix where the $ij$-th entry is obtained by taking the dot product of the $i$-th row of $A$ with the $j$-th column of $B$.

NOTE: In order to define the product of $A$ and $B$ we require that the number of columns of $A$ be equal to the number or rows of $B$. Otherwise, the product is undefined.
Linear Mappings

A **linear mapping** $F : \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$F(x) = Ax$$

for every $x$ in $\mathbb{R}^n$ where $A$ is an $m \times n$ constant matrix.

Linear mappings are **VERY** important in all applications in science and engineering and in mathematics.
Properties of linear transformations

If $F : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping then for any $x, y$ in $\mathbb{R}^n$ and any scalar $c$ we have

$$F(x + y) = F(x) + F(y)$$

$$F(c x) = c F(x).$$
Coordinate Systems

The coordinates of a point are the components of a tuple of numbers used to represent the location of the point in the plane or space.

For 2-dimensions:

- Choose an “origin” - (0,0).
- **Cartesian or rectangular coordinates** $(x, y)$

$x$-horizontal and $y$-vertical direction
Polar coordinates

\((r, \theta)\): \(r\) - distance from origin and \(\theta\) - angle from \(x\)-axis, \(0 \leq \theta < 2\pi\).

If we want to describe every point uniquely we require that \(r \geq 0\) and \(0 \leq \theta < 2\pi\).

**NOTE:** In polar coordinates you think that every point except the origin is on a circle of radius \(r\).

Polar coordinates are useful in doing computations with curves that have symmetry around the origin.
Relation between polar and cartesian coordinates

Polar to Cartesian:
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

Cartesian to Polar:
\[ r = \sqrt{x^2 + y^2} \]
\[ \theta = \tan^{-1} \left( \frac{y}{x} \right) \]
Cylindrical Coordinates

These are for 3D: \((r, \theta, z)\) and we usually think that every point in space not on the \(z\)-axis is on a cylinder.

They are good for studying objects possessing an axis of symmetry.

**Cartesian to Cylindrical**

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

**Cylindrical to Cartesian**

\[
\begin{align*}
r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right) \\
z &= z
\end{align*}
\]
Spherical Coordinates

- These coordinates are also to describe a point in 3D: \((\rho, \phi, \theta)\). They are useful to study objects that have a center of symmetry.

- Here we think as every point except \((0,0,0)\) lies on a sphere.

- \(\rho\) - distance from the origin.
  \(\phi\) - longitude and takes values \(0 \leq \phi \leq \pi\).
  \(\theta\) - latitude and takes values \(0 \leq \theta < 2\pi\).
Relation between cartesian and spherical

Spherical to cartesian:
\[ x = \rho \sin \phi \cos \theta \]
\[ y = \rho \sin \phi \sin \theta \]
\[ z = \rho \cos \phi \]

Cartesian to spherical:
\[ \rho = \sqrt{x^2 + y^2 + z^2} \]
\[ \phi = \tan^{-1}(\sqrt{x^2 + y^2}/z) \]
\[ \theta = \tan^{-1}(y/x) . \]
Relation between cylindrical and spherical

Spherical to cylindrical:
\[ r = \rho \sin \phi \]
\[ z = \rho \cos \phi \]
\[ \theta = \theta. \]

Cylindrical to spherical:
\[ \rho = \sqrt{r^2 + z^2} \]
\[ \phi = \tan^{-1}(r/z) \]
\[ \theta = \theta. \]