Functions in several variables and limits

January 9, 2006
Functions

Any function has three features:

- A **domain** set $X$;
- A **codomain** set $Y$;
- A **rule of assignment** - a rule that assign to each element $x$ in $X$ of the domain a “unique” element $f(x)$ in $Y$ (the codomain).
Scalar-valued functions

Scalar valued functions are functions such that the domain is $X \subseteq \mathbb{R}^n$ and the codomain is $\mathbb{R}$ or a subset of $\mathbb{R}$.

REMARK: Review the definitions of range, one-to-one and onto.
The Graph of a function

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a scalar valued function. Let \( x = (x_1, x_2, \ldots, x_n) \), then the graph of \( f \) is:

\[
\text{Graph} f = \{ (x, f(x)) \mid x = (x_1, \ldots, x_n) \in \mathbb{R}^n \}
\]

For example if \( f : \mathbb{R}^2 \to \mathbb{R} \), then the graph of \( f \) is the set of points in \( \mathbb{R}^3 \) that look like \( (x, y, f(x, y)) \), where \((x, y)\) is in \( \mathbb{R}^2 \).
Level Curves

Let $f$ be a function of two variables and let $c$ be a constant. The set of all $(x, y)$ in the plane such that $f(x, y) = c$ is called a level curve of $f$ with value $c$. 
Definition of the limit

**Definition:** (Intuitive) Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \), then

\[
\lim_{x \rightarrow a} f(x) = L
\]

means that we can make \( \|f(x) - L\| \) arbitrarily small (close to zero) by keeping \( \|x - a\| \) sufficiently small (but not zero).
Rigorous definition of limit

**Definition:** Let $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function. Then

$$\lim_{x \rightarrow a} f(x) = L$$

means that given $\epsilon > 0$, you can find a $\delta > 0$ (often depending on $\epsilon$) such that if $x \in X$ and $0 < \|x - a\| < \delta$, then $0 < \|f(x) - L\| < \epsilon$
Properties of limits

1. If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) then \( \lim_{x \to a} (f + g)(x) = L + M \)

2. If \( \lim_{x \to a} f(x) = L \), then \( \lim_{x \to a} kf(x) = kL \), where \( k \) is a scalar.

3. if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) then \( \lim_{x \to a} (fg)(x) = LM \)

4. If \( \lim_{x \to a} f(x) = L \) and \( g(x) \neq 0 \) for \( x \in X \), and \( \lim_{x \to a} g(x) = M \neq 0 \), then \( \lim_{x \to a} (f/g)(x) = L/M \).
Definition: Let \( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \) and let \( a \in X \). Then, \( f \) is \textbf{continuous at} \( a \) if

\[
\lim_{x \rightarrow a} f(x) = f(a).
\]

\( f \) is called \textbf{continuous} if it is continuous at every point of the domain \( X \).
• The sum \( f + g \) of two continuous functions is a continuous function.

• The scalar multiple of a continuous function \( kf \) is continuous.

• The product \( fg \) and the quotient \( f/g \) (when defined) of two continuous functions is continuous.