Derivatives

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Partial Derivative

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation. Let $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ then the partial derivative with respect to $x_i$ is:

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

we also use $f_{x_i}$ for partial derivative.
Tangent Planes

Let \( f : X \subseteq \mathbb{R}^2 \to \mathbb{R} \). If the graph of \( z = f(x, y) \) has a tangent plane at \((a, b, f(a, b))\), then the tangent plane has equation

\[
z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]
REMARK: The existence of a tangent plane to the graph of \( z = f(x, y) \) is a stronger condition than the existence of partial derivatives.

\[
f(x, y) = ||x| - |y|| - |x| - |y|
\]

is a function with partial derivatives at (0,0), but no tangent plane at (0,0).
Good Linear Approximation

We say that

\[ h(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \]

is a **good linear approximation** to the function \( f : X \subset \mathbb{R}^2 \to \mathbb{R} \) at the point \((a, b)\) if

\[
\lim_{(x, y) \to (a, b)} \frac{f(x, y) - h(x, y)}{\| (x, y) - (a, b) \|} = 0
\]
Differentiable

A function $f : X \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at $(a, b) \in X$ if

(1) the partial derivatives $f_x$ and $f_y$ exist at $(a, b)$.

(2) $h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ is a good linear approximation of $f(x, y)$ near $(a, b)$. 
A function that is differentiable at all points in the domain is called **differentiable**.

**NOTE:** We require that $X$ be an open set.
Generalization to $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

$$h(x) = f(a) + f_{x_1}(a)(x_1-a_1) + \cdots + f_{x_n}(a)(x_n-a_n)$$

is the generalization to the tangent plane. We say that $h(x)$ is a good linear approximation to $f(x, y)$ near $a$ if

$$\lim_{x \to a} \frac{f(x) - h(x)}{x - a} = 0$$
Differentiability of $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

We say that $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is \textbf{differentiable} at $a$ if

(1) all partial derivatives $f_{x_i}$ exist at $a$.

(2) $h(x)$ is a good linear approximation to $f(x)$ near $a$.

We say that $f$ is \textbf{differentiable} if $f$ is differentiable at every point in the domain $X$ (open set).
The Gradient of $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

The **gradient** of $f$ is

$$\nabla f(x) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

$h(x) = f(a) + f_{x_1}(a)(x_1-a_1) + \cdots + f_{x_n}(a)(x_n-a_n)$

can be rewritten

$$h(x) = f(a) + \nabla f(a) \cdot (x - a)$$

Here we think of $\nabla f(a)$ and $x - a$ as vectors.
Derivative Matrix for scalar valued functions

\[ Df(a) = \begin{bmatrix} f_{x_1}(a) & f_{x_2}(a) & \cdots & f_{x_n}(a) \end{bmatrix} \]

This is a $1 \times n$ matrix.

We can rewrite,

\[ \nabla f(a) \cdot (x - a) = Df(a)(x - a) \]

in the right-handside we think of $(x - a)$ as a $n \times 1$ vector.
General Derivative Matrix

Let \( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a function \( f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \)

\[
Df(x) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{pmatrix}
\]
Grand Definition of Differentiability

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $a$ in $X$. $f$ is differentiable at $a$ if

(1) $Df(a)$ exists and

(2) $h(x) = f(a) + Df(a)(x - a)$ is a good linear approximation to $f$ near $a$. 
Properties about the derivative

Let $f$ and $g$ be two differentiable functions then

(1) $D(f + g)(a) = D(f)(a) + D(g)(a)$

(2) $D(cf)(a) = cDf(a)$ for any scalar $c$.

If $f$ and $g$ are scalar valued functions:

(1) $D(fg)(a) = g(a)Df(a) + f(a)Dg(a)$. 

(2) $D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2}$. 