Chain Rule

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The chain rule in one variable

Suppose that \( x \) is differentiable at \( t_0 \) and \( f \) is differentiable at \( x_0 \), then the composite function \( f \circ x \) is differentiable at \( t_0 \) and, moreover,

\[
(f \circ x)'(t_0) = f'(x_0)x'(t_0)
\]

or

\[
\frac{df}{dt}(t_0) = \frac{df}{dx}(x_0) \frac{dx}{dt}(t_0)
\]
The chain rule in two variables

Let $f : X \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at $x_0 = (x_0, y_0)$ and $x : T \subseteq \mathbb{R} \rightarrow \mathbb{R}^2$ is differentiable at $t_0$. Then

$$\frac{df}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0) \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0) \frac{dy}{dt}(t_0)$$

can be rewritten:

$$\frac{df}{dt}(t_0) = \left( \frac{\partial f}{\partial x}(x_0), \frac{\partial f}{\partial y}(x_0) \right) \cdot \left( \frac{dx_1}{dt}(t_0), \frac{dx_2}{dt}(t_0) \right)$$
Generalization to functions \( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \)

Let \( x : T \subseteq \mathbb{R} \rightarrow \mathbb{R}^n \)

\[
\frac{df}{dt}(t_0) = Df(x_0)Dx(t_0) \\
= \nabla f(x_0) \cdot x'(t_0)
\]
Generalization when $x$ is a surface

$f : X \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ and $x : T \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$

\[
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}
\]

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}
\]
The general chain rule

\[ f : X \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^p \text{ and } x : T \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m. \]

\[ D(f \circ x)(t_0) = Df(x_0)Dx(t_0) \]