The Gradient and Directional Derivatives

January 11, 2006
The gradient

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar valued function. Then the gradient

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$
Directional Derivative

Consider a scalar-valued function $f$, a point $a$ in the domain of $f$ and $v$ any unit vector then the **directional derivative of $f$ in the direction of $v$**, denoted $D_v f(a)$, is

$$D_v f(a) = \lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}$$

provided the limit exists.
Computing the directional derivative using the gradient

Let $f$ be a differentiable function and $a$ be a point in the domain of $f$ then

$$D_v f(a) = \nabla f(a) \cdot v,$$

where $v$ is a unit vector.
Maximum and minimum values of $D_v f(a)$

- $D_v f(a)$ is maximized when $v$ points in the same direction of the gradient, $\nabla f(a)$.

- $D_v f(a)$ is minimized when $v$ points in the opposite direction of the gradient, $-\nabla f(a)$.

- Furthermore, the maximum and minimum values of $D_v f(a)$ are $\|\nabla f(a)\|$ and $-\|\nabla f(a)\|$, respectively.
Tangent planes to level surfaces: \( f(x) = c \)

Let \( c \) be any constant.

If \( x_0 \) is a point on the level surface \( f(x) = c \), then the vector \( \nabla f(x_0) \) is perpendicular to the surface at \( x_0 \).
Computing Tangent plane for level surfaces

Given the equation of a level surface \( f(x, y, z) = c \) and a point \( x_0 \), then the equation of the tangent plane is

\[
\nabla f(x_0) \cdot (x - x_0) = 0
\]

or if \( x_0 = (x_0, y_0, z_0) \) then

\[
f_x(x_0)(x-x_0) + f_y(x_0)(y-y_0) + f_z(x_0)(z-z_0) = 0.
\]