More Matrices

**Problem 1.** Let $A$ be an $m \times n$ matrix. Show that if $Ax = 0$ for every $x \in \mathbb{R}^n$ then $A$ is the zero matrix (that is, every entry in $A$ is 0). *Hint:* What happens if you take $x = e_i$?

**Problem 2.** Let

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be a symmetric $2 \times 2$ matrix with $a \neq 0$. As in the text, define the associated quadratic form by

$$Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} B \begin{pmatrix} x \\ y \end{pmatrix}$$

(1)

$$= ax^2 + bxy + cy^2$$

(2)

$$= a \left( \left( x + \frac{b}{a} y \right)^2 + \frac{ac - b^2}{a^2} y^2 \right).$$

(3)

If $ac - b^2 < 0$ show that this form is *indefinite*; that is, show that $Q$ takes on both positive and negative values. In fact, show that there are arbitrarily small points where $Q$ is positive and arbitrarily small points where $Q$ is negative. *Hint:* Try to find small points that make one of the two terms in (3) vanish.

**Problem 3.** Use Problem 2 and the Second Order Taylor Approximation to show that if $f : \mathbb{R}^2 \to \mathbb{R}$ is of class $C^3$, $x_0$ is a critical point of $f$ and the Hessian $Hf(x_0)$ is indefinite, then $f$ has a saddle point at $x_0$. That is, show that under these hypotheses there are points $x$ arbitrarily close to $x_0$ for which $f(x) > f(x_0)$ and points $x$ arbitrarily close to $x_0$ for which $f(x) < f(x_0)$. 