Ergodic Markov Chains

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Definition

- A Markov chain is called an **ergodic chain** if it is possible to go from every state to every state (not necessarily in one move).

- Ergodic Markov chains are also called **irreducible**.

- A Markov chain is called a **regular chain** if some power of the transition matrix has only positive elements.
Example

• Let the transition matrix of a Markov chain be defined by

\[
P = \begin{pmatrix} 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.
\]

• Then this is an ergodic chain which is not regular.
Example: Ehrenfest Model

- We have two urns that, between them, contain four balls.

- At each step, one of the four balls is chosen at random and moved from the urn that it is in into the other urn.

- We choose, as states, the number of balls in the first urn.

\[
P = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1/4 & 0 & 3/4 & 0 & 0 \\
2 & 0 & 1/2 & 0 & 1/2 & 0 \\
3 & 0 & 0 & 3/4 & 0 & 1/4 \\
4 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]
Regular Markov Chains

• Any transition matrix that has no zeros determines a regular Markov chain.

• However, it is possible for a regular Markov chain to have a transition matrix that has zeros.

• For example, recall the matrix of the Land of Oz

\[
P = \begin{pmatrix}
R & 1/2 & 1/4 & 1/4 \\
N & 1/2 & 0 & 1/2 \\
S & 1/4 & 1/4 & 1/2 \\
\end{pmatrix}.
\]
Theorem. Let $P$ be the transition matrix for a regular chain. Then, as $n \to \infty$, the powers $P^n$ approach a limiting matrix $W$ with all rows the same vector $w$. The vector $w$ is a strictly positive probability vector (i.e., the components are all positive and they sum to one).
Example

• For the Land of Oz example, the sixth power of the transition matrix $P$ is, to three decimal places,

$$P^6 = \begin{pmatrix} R & N & S \\ R & 0.4 & 0.2 & 0.4 \\ N & 0.4 & 0.2 & 0.4 \\ S & 0.4 & 0.2 & 0.4 \end{pmatrix}.$$
Theorem. Let $P$ be a regular transition matrix, let

$$W = \lim_{n \to \infty} P^n,$$

let $w$ be the common row of $W$, and let $c$ be the column vector all of whose components are 1. Then

(a) $wP = w$, and any row vector $v$ such that $vP = v$ is a constant multiple of $w$.

(b) $Pc = c$, and any column vector $x$ such that $Px = x$ is a multiple of $c$. 

Definition: Fixed Vectors

- A row vector $w$ with the property $wP = w$ is called a fixed row vector for $P$.

- Similarly, a column vector $x$ such that $Px = x$ is called a fixed column vector for $P$.
Example

Find the limiting vector $\mathbf{w}$ for the Land of Oz.
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\[ w_1 + w_2 + w_3 = 1 \]

and

\[
\begin{pmatrix}
  w_1 & w_2 & w_3 \\
\end{pmatrix}
\begin{pmatrix}
  1/2 & 1/4 & 1/4 \\
  1/2 & 0 & 1/2 \\
  1/4 & 1/4 & 1/2 \\
\end{pmatrix}
= \begin{pmatrix}
  w_1 & w_2 & w_3 \\
\end{pmatrix}.
\]
Example

Find the limiting vector $\mathbf{w}$ for the Land of Oz.

$$w_1 + w_2 + w_3 = 1$$

and

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}.$$
\[ w_1 + w_2 + w_3 = 1 , \]
\[ (1/2)w_1 + (1/2)w_2 + (1/4)w_3 = w_1 , \]
\[ (1/4)w_1 + (1/4)w_3 = w_2 , \]
\[ (1/4)w_1 + (1/2)w_2 + (1/2)w_3 = w_3 . \]
\[ w_1 + w_2 + w_3 = 1, \]
\[ (1/2)w_1 + (1/2)w_2 + (1/4)w_3 = w_1, \]
\[ (1/4)w_1 + (1/4)w_3 = w_2, \]
\[ (1/4)w_1 + (1/2)w_2 + (1/2)w_3 = w_3. \]

The solution is
\[ \mathbf{w} = (0.4 \quad 0.2 \quad 0.4), \]
Another method

• Assume that the value at a particular state, say state one, is 1, and then use all but one of the linear equations from $wP = w$.

• This set of equations will have a unique solution and we can obtain $w$ from this solution by dividing each of its entries by their sum to give the probability vector $w$. 
Example (cont’d)

• Set $w_1 = 1$, and then solve the first and second linear equations from $wP = w$.

\[
\begin{align*}
(1/2) + (1/2)w_2 + (1/4)w_3 &= 1, \\
(1/4) + (1/4)w_3 &= w_2.
\end{align*}
\]

• We obtain

\[
\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 1 \end{pmatrix}.
\]
Equilibrium

• Suppose that our starting vector picks state $s_i$ as a starting state with probability $w_i$, for all $i$.

• Then the probability of being in the various states after $n$ steps is given by $wP^n = w$, and is the same on all steps.

• This method of starting provides us with a process that is called “stationary.”
Ergodic Markov Chains

**Theorem.** For an ergodic Markov chain, there is a unique probability vector \( \mathbf{w} \) such that \( \mathbf{wP} = \mathbf{w} \) and \( \mathbf{w} \) is strictly positive. Any row vector such that \( \mathbf{vP} = \mathbf{v} \) is a multiple of \( \mathbf{w} \). Any column vector \( \mathbf{x} \) such that \( \mathbf{Px} = \mathbf{x} \) is a constant vector.
The Ergodic Theorem

**Theorem.** Let $P$ be the transition matrix for an ergodic chain. Let $A_n$ be the matrix defined by

$$A_n = \frac{I + P + P^2 + \ldots + P^n}{n + 1}.$$ 

Then $A_n \to W$, where $W$ is a matrix all of whose rows are equal to the unique fixed probability vector $w$ for $P$. 
Exercises

Which of the following matrices are transition matrices for regular Markov chains?

1. \( P = \begin{pmatrix} .5 & .5 \\ 1 & 0 \end{pmatrix} \).

2. \( P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{pmatrix} \).

3. \( P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).
Consider the Markov chain with general $2 \times 2$ transition matrix

$$P = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}.$$ 

1. Under what conditions is $P$ absorbing?

2. Under what conditions is $P$ ergodic but not regular?

3. Under what conditions is $P$ regular?
Find the fixed probability vector $\mathbf{w}$ for the matrices in the previous exercise that are ergodic.