Problem

Show that

\[ b(n, p, j) = \frac{p}{q} \left( \frac{n - j + 1}{j} \right) b(n, p, j - 1) , \]

for \( j \geq 1 \). Use this fact to determine the value or values of \( j \) which give \( b(n, p, j) \) its greatest value.
Problem

Show that the number of ways that one can put $n$ different objects into three boxes with $a$ in the first, $b$ in the second, and $c$ in the third is $n!/(a!b!c!)$. 
Problem

Prove that the probability of exactly $n$ heads in $2n$ tosses of a fair coin is given by the product of the odd numbers up to $2n - 1$ divided by the product of the even numbers up to $2n$. 
Conditional Probability

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Example

Three candidates A, B, and C are running for office. We decided that A and B have an equal chance of winning and C is only 1/2 as likely to win as A. Let $A$ be the event “$A$ wins,” $B$ that “$B$ wins,” and $C$ that “$C$ wins.” Hence, we assigned probabilities $P(A) = \frac{2}{5}$, $P(B) = \frac{2}{5}$, and $P(C) = \frac{1}{5}$.

Suppose that before the election is held, $A$ drops out of the race. What are the values for $P(B | A)$ and $P(C | A)$?
Definition

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_r\}$ be the original sample space with distribution function $m(\omega_j)$ assigned. Suppose we learn that the event $E$ has occurred.

- If a sample point $\omega_j$ is not in $E$, we want $m(\omega_j|E) = 0$.
- For $\omega_k$ in $E$, we should have the same relative magnitudes that they had before we learned that $E$ had occurred:

$$m(\omega_k|E) = cm(\omega_k).$$
But we must also have

$$\sum_{E} m(\omega_k|E) = c \sum_{E} m(\omega_k) = 1.$$ 

Thus,

$$c = \frac{1}{\sum_{E} m(\omega_k)} = \frac{1}{P(E)}.$$
**Definition 1.** The conditional distribution given $E$ is the distribution on $\Omega$ defined by

$$m(\omega_k | E) = \frac{m(\omega_k)}{P(E)}$$

for $\omega_k$ in $E$, and $m(\omega_k | E) = 0$ for $\omega$ not in $E$. 
Then, for a general event $F$,

$$P(F|E) = \sum_{F \cap E} m(\omega_k|E) = \sum_{F \cap E} \frac{m(\omega_k)}{P(E)} = \frac{P(F \cap E)}{P(E)}.$$ 

We call $P(F|E)$ the conditional probability of $F$ occurring given that $E$ occurs.
Example

Let us return to the example of rolling a die. Recall that $F$ is the event $X = 6$, and $E$ is the event $X > 4$. Note that $E \cap F$ is the event $F$. So, the above formula gives

$$P(F|E) = \frac{P(F \cap E)}{P(E)}$$

$$= \frac{1/6}{1/3}$$

$$= \frac{1}{2}.$$
Example

We have two urns, I and II. Urn I contains 2 black balls and 3 white balls. Urn II contains 1 black ball and 1 white ball. An urn is drawn at random and a ball is chosen at random from it. We can represent the sample space of this experiment as the paths through a tree.
\[
\begin{array}{cccc}
\text{Urn} & \text{Color of ball} & \omega & p(\omega) \\
\text{I} & 2/5 & b & \omega_1 & 1/5 \\
 & 3/5 & w & \omega_2 & 3/10 \\
\text{II} & 1/2 & b & \omega_3 & 1/4 \\
 & 1/2 & w & \omega_4 & 1/4 \\
\end{array}
\]
• Let $B$ be the event “a black ball is drawn,” and $I$ the event “urn I is chosen.” Then the branch weight $2/5$, which is shown on one branch in the figure, can now be interpreted as the conditional probability $P(B|I)$.

• What is $P(I|B)$?
Bayes Probabilities

We have just calculated the inverse probability that a particular urn was chosen, given the color of the ball. Such an inverse probability is called a Bayes probability.
<table>
<thead>
<tr>
<th>Color of ball</th>
<th>Urn</th>
<th>$\omega$</th>
<th>$p(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>$\omega_1$</td>
<td>1/5</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>$\omega_3$</td>
<td>1/4</td>
</tr>
<tr>
<td></td>
<td>I</td>
<td>$\omega_2$</td>
<td>3/10</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>$\omega_4$</td>
<td>1/4</td>
</tr>
</tbody>
</table>

(start)

- **B**: 
  - Urn I: $\omega_1$, $p(\omega_1) = 1/5$
  - Urn II: $\omega_3$, $p(\omega_3) = 1/4$

- **W**: 
  - Urn I: $\omega_2$, $p(\omega_2) = 3/10$
  - Urn II: $\omega_4$, $p(\omega_4) = 1/4$
The Monty Hall problem

Suppose you’re on Monty Hall’s *Let’s Make a Deal!* You are given the choice of three doors, behind one door is a car, the others, goats. You pick a door, say 1, Monty opens another door, say 3, which has a goat. Monty says to you “Do you want to pick door 2?” Is it to your advantage to switch your choice of doors?

**Question:** What is the conditional probability that you win if you switch, given that you have chosen door 1 and that Monty has chosen door 3.
Door opened by Monty

Path probabilities

Placement of car

Door chosen by contestant

1/18

1/9

1/9

1/9
Problem

Assume that $E$ and $F$ are two events with positive probabilities. Show that if $P(E|F) = P(E)$, then $P(F|E) = P(F)$. 
Problem

A die is rolled twice. What is the probability that the sum of the faces is greater than 7, given that

1. the first outcome was a 4?
2. the first outcome was greater than 3?
3. the first outcome was a 1?
4. the first outcome was less than 5?