Important Distributions

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Discrete Uniform Distribution

• All outcomes of an experiment are equally likely.

• If $X$ is a random variable which represents the outcome of an experiment of this type, then we say that $X$ is uniformly distributed.

• If the sample space $S$ is of size $n$, where $0 < n < \infty$, then the distribution function $m(\omega)$ is defined to be $1/n$ for all $\omega \in S$. 
Binomial Distribution

- The distribution of the random variable which counts the number of heads which occur when a coin is tossed \( n \) times, assuming that on any one toss, the probability that a head occurs is \( p \).

- The distribution function is given by the formula

\[
b(n, p, k) = \binom{n}{k} p^k q^{n-k},
\]

where \( q = 1 - p \).
Exercise

A die is rolled until the first time $T$ that a six turns up.

1. What is the probability distribution for $T$?

2. Find $P(T > 3)$.

3. Find $P(T > 6 | T > 3)$. 
Geometric Distribution

- Consider a Bernoulli trials process continued for an infinite number of trials; for example, a coin tossed an infinite sequence of times.

- Let $T$ be the number of trials up to and including the first success. Then

\[
P(T = 1) = p,
\]
\[
P(T = 2) = qp,
\]
\[
P(T = 3) = q^2p,
\]

and in general,

\[
P(T = n) = q^{n-1}p.
\]
Exercise

Cards are drawn, one at a time, from a standard deck; each card is replaced before the next one is drawn. Let $X$ be the number of draws necessary to get an ace. Find $E(X)$. 
Example

Suppose a line of customers waits for service at a counter. It is often assumed that, in each small time unit, either 0 or 1 new customers arrive at the counter. The probability that a customer arrives is \( p \) and that no customer arrives is \( q = 1 - p \). Let \( T \) be the time until the next arrival. What is the probability that no customer arrives in the next \( k \) time units, that is, for \( P(T > k) \).
Negative Binomial Distribution

- Suppose we are given a coin which has probability $p$ of coming up heads when it is tossed.

- We fix a positive integer $k$, and toss the coin until the $k$th head appears.

- Let $X$ represent the number of tosses. When $k = 1$, $X$ is geometrically distributed.

- For a general $k$, we say that $X$ has a negative binomial distribution.

- What is the probability distribution $u(x, k, p)$ of $X$?
Example

A fair coin is tossed until the second time a head turns up. The distribution for the number of tosses is \( u(x, 2, p) \). What is the probability that \( x \) tosses are needed to obtain two heads.
The Poisson Distribution

• The Poisson distribution can be viewed as arising from the binomial distribution, when \( n \) is large and \( p \) is small.

• The Poisson distribution with parameter \( \lambda \) is obtained as a limit of binomial distributions with parameters \( n \) and \( p \), where it was assumed that \( np = \lambda \), and \( n \to \infty \).

\[
P(X = k) \approx \frac{\lambda^k}{k!} e^{-\lambda}.
\]
Example

- A typesetter makes, on the average, one mistake per 1000 words. Assume that he is setting a book with 100 words to a page.

- Let $S_{100}$ be the number of mistakes that he makes on a single page.

- Then the exact probability distribution for $S_{100}$ would be obtained by considering $S_{100}$ as a result of 100 Bernoulli trials with $p = 1/1000$.

- The expected value of $S_{100}$ is $\lambda = 100(1/1000) = .1$. 
The exact probability that $S_{100} = j$ is $b(100, 1/1000, j)$, and the Poisson approximation is

$$e^{-\lambda} \left(\frac{\lambda}{j!}\right)^j.$$
The Poisson distribution with parameter $\lambda = .3$ has been assigned for the outcome of an experiment. Let $X$ be the outcome function. Find $P(X = 0)$, $P(X = 1)$, and $P(X > 1)$. 

Exercise
In a class of 80 students, the professor calls on 1 student chosen at random for a recitation in each class period. There are 32 class periods in a term.

1. Write a formula for the exact probability that a given student is called upon \( j \) times during the term.

2. Write a formula for the Poisson approximation for this probability. Using your formula estimate the probability that a given student is called upon more than twice.
Hypergeometric Distribution

- Suppose that we have a set of $N$ balls, of which $k$ are red and $N - k$ are blue.

- We choose $n$ of these balls, without replacement, and define $X$ to be the number of red balls in our sample.

- The distribution of $X$ is called the hypergeometric distribution.

- Note that this distribution depends upon three parameters, namely $N$, $k$, and $n$. 
• We will use the notation $h(N, k, n, x)$ to denote $P(X = x)$.

• The distribution function is

$$h(N, k, n, x) = \binom{k}{x} \binom{N-k}{n-x} \binom{N}{n}.$$
Example

A bridge deck has 52 cards with 13 cards in each of four suits: spades, hearts, diamonds, and clubs. A hand of 13 cards is dealt from a shuffled deck. Find the probability that the hand has

1. a distribution of suits 4, 4, 3, 2 (for example, four spades, four hearts, three diamonds, two clubs).

2. a distribution of suits 5, 3, 3, 2.