1. (20) Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^5$ defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + x_3, -3x_1 + 2x_2 - x_4, 3x_2 + 9x_3 + 3x_4).$$

Let $A$ be the matrix of $T$ (i.e., $T(x) = Ax$).

(i) Find $A$. (A mistake here will affect the rest of the problem.)

(ii) Find a basis for $\text{Col } A$.

(iii) Find a basis for $\text{Row } A$.

(iv) What is the dimension of the kernel of $T$? (No details necessary.)

(v) Is $T$ onto? Give a reason for your answer.

2. (20) Let $V$ be a two dimensional vector space with basis $B = \{v_1, v_2\}$. Let $a$ be a fixed scalar and let $T : V \to V$ be a linear transformation such that $T(v_1) = av_2$ and $T(v_2) = av_1$.

(i) What is the matrix $[T]_B$?

(ii) Is $[T]_B$ diagonalizable? Give reasons for your answer.

(iii) If $v \in V$ and $[v]_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, what is $[T(v)]_B$?

3. (20) Find a basis for all vectors of the form

$$\begin{pmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$. 

\[\text{the subspace of } \mathbb{R}^4\]
4. (20) Consider the matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & -2 \\
-1 & 1 & 3 \\
\end{pmatrix}
\]
Find all eigenvalues and a basis for each eigenspace.

5. (30) True - False. In each of the following, circle T if the statement is always true; circle F otherwise.

(a). If \(\{v_1, \ldots, v_k\}\) is a linearly independent set of vectors in a vector space \(V\), then every vector in \(\text{Span}\{v_1, \ldots, v_k\}\) can be written in exactly one way as a linear combination of \(v_1, \ldots, v_k\).

(b). If the \(n \times n\) matrices \(A\) and \(B\) are both similar to an \(n \times n\) matrix \(C\), then \(A\) is similar to \(B\).

(c). \(\text{Col} \ A\) is the set of all vectors that can be written as \(Ax\) for some \(x\).

(d). If the nullspace of a \(5 \times 6\) matrix \(A\) is 4-dimensional, then \(\text{Col} \ A\) is 1-dimensional.

(e). \(\text{Col} \ A = \text{Row} \ A^T\), for any matrix \(A\).

(f). If \(A\) is a \(7 \times 5\) matrix, then the largest possible rank of \(A\) is 5.

(g). If 0 is an eigenvalue of an \(n \times n\) matrix \(A\), then \(A\) is not invertible.

(h). If a \(4 \times 4\) matrix \(A\) has exactly 3 distinct eigenvalues, then \(A\) is not diagonalizable.

(i). The set of all eigenvectors of an \(n \times n\) matrix \(A\) is a subspace of \(\mathbb{R}^n\).

(j). If \(A\), \(P\) and \(D\) are \(n \times n\) matrices such that \(P\) is invertible, \(D\) is diagonal and \(A = PDP^{-1}\) then the columns of \(P\) are eigenvectors of \(A\).