Homework Problems

November 2, 2005

Exercise 1. Let $D : \mathbb{P}_4 \rightarrow \mathbb{P}_4$ be the linear transformation given by

$$D(p(t)) = (1 - t^2)p''(t) - 2tp'(t) + 20p(t).$$

a. Using coordinates, find bases for the kernel and range of $D$.

b. Use the result of part (a) to conclude that, up to scalar multiples, there is only one polynomial solution of degree $\leq 4$ to the differential equation

$$(1 - t^2)p'' - 2tp' + 20p = 0.$$ What is this solution?

c. Use the result of part (a) to produce a polynomial $q$ of degree at most 4 so that the differential equation

$$(1 - t^2)p'' - 2tp' + 20p = q$$

has no polynomial solution of degree $\leq 4$.

Recall the following fact from elementary algebra.

**Theorem 1.** If $p(t)$ is a polynomial with real coefficients and $a$ is a real number with $p(a) = 0$ then there is a polynomial $q(t)$ with real coefficients so that $p(t) = (t - a)q(t)$.

In the next exercise we will provide a linear algebraic proof of this fact, at least for polynomials of degree $\leq 3$.

Exercise 2. Fix a real number $a$ and consider the linear transformation $T : \mathbb{P}_3 \rightarrow \mathbb{R}$ given by

$$T(p) = p(a).$$

a. Using coordinates relative to the bases $B = \{1, t, t^2, t^3\}$ and $E = \{1\}$, find bases for the kernel and range of $T$.

b. Show directly (without using Theorem 1 above) that the polynomials in your basis for ker $T$ are all divisible by $t - a$. Conclude that all the polynomials in ker $T$ are divisible by $t - a$.

c. Show that part (b) proves Theorem 1 for polynomials of degree $\leq 3$?

Exercise 3. [Extra Credit] Apply the technique of Exercise 2 to the linear transformation $T : \mathbb{P}_n \rightarrow \mathbb{R}$ given by $T(p) = p(a)$ to prove Theorem 1 in general.