Workshop 5
Subspaces

Instructions:

Get into groups and work on the following exercises. Each group is expected to turn in one neatly written copy of their solutions at the end of the class period.

Exercise 1. Let \( V \) be a vector space and let \( H \subset V \) be a subspace. Show that if \( u \) and \( v \) are two vectors in \( H \), then \( \text{Span}\{u, v\} \) is contained in \( H \). Can you generalize this statement?

Exercise 2. Let \( V \) be a vector space and let \( H, K \subset V \) be subspaces. The intersection of \( H \) and \( K \), denoted \( H \cap K \), is the collection of all vectors that belong to both \( H \) and \( K \) simultaneously. In set notation
\[
H \cap K = \{v : v \text{ is in both } H \text{ and } K\}.
\]
Show that \( H \cap K \) is a subspace of \( V \).

Exercise 3. Let \( V \) be a vector space and let \( H, K \subset V \) be subspaces. The sum of \( H \) and \( K \), denoted \( H + K \), is the collection of all vectors of the form \( u + v \) where \( u \in H \) and \( v \in K \). In set notation
\[
H + K = \{w : w = u + v, \ u \in H, \ v \in K\}.
\]
Show that \( H + K \) is a subspace of \( V \).

Exercise 4.* Let \( V \) and \( W \) be vector spaces and let \( T : V \to W \) be a linear transformation. Let \( H \) be a subspace of \( W \) and let \( T^{-1}(H) \) denote the set of all vectors \( v \in V \) so that \( T(v) \in H \). In set notation
\[
T^{-1}(H) = \{v \in V : T(v) \in H\}.
\]
Show that \( T^{-1}(H) \) is a subspace of \( V \).\(^1\)

\(^1\)The notation \( T^{-1}(H) \) is purely symbolic. It does not mean that \( T \) is in invertible.