Workshop 8
Rank

Instructions:

Get into groups and work on the following exercises. Each group is expected to turn in one neatly written copy of their solutions at the end of the class period.

Exercise 1. Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times p$ matrix.

a. Show that $\text{Nul } B \subset \text{Nul } AB$. Conclude that $\dim \text{Nul } B \leq \dim \text{Nul } AB$.

b. Show that $\text{Col } AB \subset \text{Col } A$. Conclude that $\text{rank } AB \leq \text{rank } A$.

c. Use parts (a) and (b) together with the rank theorem to show that

$$\text{rank } AB \leq \min \{ \text{rank } A, \text{rank } B \}.$$ 

That is, show that $\text{rank } AB \leq \text{rank } A$ and $\text{rank } AB \leq \text{rank } B$. [Hint: Write down the conclusion of the rank theorem for each of $A$, $B$, and $AB$ and compare.]

Exercise 2. Use the results of Problem 2 to show that if $A$ and $B$ are both $n \times n$ then

$$\dim \text{Nul } AB \geq \max \{ \dim \text{Nul } A, \dim \text{Nul } B \}.$$ 

That is, show that $\dim \text{Nul } AB \geq \dim \text{Nul } A$ and $\dim \text{Nul } AB \geq \dim \text{Nul } B$. [Hint: See the hint for the previous problem.]

Exercise 3. Let $A$ be an $m \times n$ matrix and let $A^T$ be its transpose, which is an $n \times m$ matrix.

a. Use the rank theorem to show that

$$\dim \text{Col } A + \dim \text{Nul } A^T = m.$$ 

b. Use part (a) to show that $Ax = b$ has a solution for every $b \in \mathbb{R}^m$ if and only if $A^T y = 0$ has only the trivial solution.

c. If $A$ is square (i.e. $m = n$) use part (a) to show that $A$ is invertible if and only if $A^T$ is invertible.

Exercise 4.* If $A$ is an $m \times n$ matrix and rank $A = 1$, show that there are vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ so that $A = uv^T$. [Hint: Show that all of the columns of $A$ are multiples of one another.]