Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with the property that $T \circ T \circ T = 0$ (we’ll refer to $T \circ T \circ T$ as $T^3$ for the rest of this problem).

(a) What exactly does this mean? That is, what is the practical upshot when it comes to plugging in vectors to $T^3$?

Solution

It means that $T^3(v) = 0$ for every vector $v \in \mathbb{R}^3$.

(b) Suppose that $x \in \mathbb{R}^3$ is such that $T^2(x) = T(T(x)) \neq 0$. If $z = cT^2(x)$, then what is $T(z)$? Let $y = c_1T(x) + c_2T^2(x)$. What is $T^2(y)$?

Solution

Well, 

$$T(z) = T(cT^2(x)) = cT^3(x) = 0$$

and

$$T^2(y) = T^2(c_1T(x) + c_2T^2(x)) = c_1T^3(x) + c_2T^3(T(x)) = 0.$$ 

(c) Let’s say

$$b_1 = x \quad b_2 = T(x), \quad \text{and} \quad b_3 = T^2(x).$$

Show that $b_1$ is not a linear combination of $b_2$ and $b_3$.

Solution

From what we’ve seen in (b), any linear combination $y$ of $b_2$ and $b_3$ has the property that $T^2(y) = 0$. Since $T^2(b_1) = T^2(x) \neq 0$, $b_1$ can’t be a linear combination of $b_2$ and $b_3$. 

1
(d) Explain why the set $\mathcal{B} = \{b_1, b_2, b_3\}$ is a basis for $\mathbb{R}^3$. (Hint: Some of the work you’ve already done might help.)

Solution

We know that $T(b_2) = T^2(x) \neq 0$ and $T(b_3) = 0$, so $b_2$ is not a multiple of $b_1$. And we showed in (c) that $b_1$ is not a linear combination of $b_2$ and $b_3$, so the whole set must be linearly independent. Since we are working in $\mathbb{R}^3$, any linearly independent set of 3 vectors is a basis.

(e) Find the $\mathcal{B}$-matrix for the linear transformation $T$. (This can be done with very little work).

Solution

Since $T(b_1) = b_2$, $T(b_2) = b_3$ and $T(b_3) = 0$, the matrix must be

\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

2 Let

\[
A = \begin{pmatrix}
3 & 0 & 0 \\
0 & 4 & 1 \\
0 & 2 & 5
\end{pmatrix}.
\]

(a) Find the eigenvalues of $A$. (Hint: $\lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6)$)

Solution

The characteristic polynomial is

\[
\det(A-\lambda I) = (3-\lambda)((4-\lambda)(5-\lambda)-2) = (3-\lambda)(\lambda^2-9\lambda+18) = (3-\lambda)(\lambda-3)(\lambda-6).
\]

Therefore the eigenvalues of $A$ are 3 and 6.
(b) Find bases for the eigenspaces of $A$.

Solution

First let’s handle $\lambda = 3$. We need to solve $(A - 3I)v = 0$ by reducing the matrix

$$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that the eigenspace for $\lambda = 3$ is

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

For $\lambda = 6$, we row reduce

$$A - 6I = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that the eigenspace for $\lambda = 3$ is

$$\text{span}\left\{ \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} \right\}.$$

(c) Write down an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. Briefly explain yourself.

Solution

We have a basis

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$$
for $\mathbb{R}^3$ consisting of eigenvectors for $A$. The matrix $P$ is the change of coordinates from $\mathcal{B}$ to the standard basis, and $D$ is the diagonal matrix with the eigenvalues on the diagonal. Thus if

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix},$$

then $A = PDP^{-1}$.

3

(a) Suppose $T : V \rightarrow V$ is a linear transformation, and that $\mathcal{B} = \{b_1, b_2, b_3\}$ is a basis for $V$. If the $\mathcal{B}$-matrix for $T$ is

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{pmatrix},$$

then what is $T(2b_1 + 4b_3)$?

Solution

Calculate $T(2b_1 + 4b_3) = 16b_1 + 58b_2 + 110b_3$.

(b) Explain why the image (range) of a linear transformation $T : V \rightarrow W$ is a subspace of $W$.

Solution

The image of $T$ is the set $\text{Im}(T) = \{T(v) \mid v \in V\}$. If $x, y \in \text{Im}(T)$ and $c, d \in \mathbb{R}$, then $x = T(v)$ for some $v \in V$ and $y = T(u)$ for some $u \in V$. Then

$$cx + dy = cT(v) + dT(u) = T(cv + du).$$

Since $V$ is a vector space, $cv + du \in V$, so $cx + dy \in \text{Im}(T)$.
(c) Is the matrix
\[
A = \begin{pmatrix}
1 & 2 & 3 \\
0 & 5 & 8 \\
0 & 0 & 13
\end{pmatrix}
\]
diagonalizable? Explain.

**Solution**

Yes. It is upper triangular, so we can get the eigenvalues from the diagonal. Since there are three distinct eigenvalues and we’re working with \( \mathbb{R}^3 \), there will be a basis for \( \mathbb{R}^3 \) consisting of eigenvectors of \( A \), which is the same as saying that \( A \) is diagonalizable.

(d) If the column space of a \( 8 \times 4 \) matrix \( A \) is 3-dimensional, then what is the dimension of the null space?

**Solution**

There are 3 pivot columns, leaving 1 nonpivot column, so the dimension of \( \text{Nul}(A) \) is 1.

4 Consider the matrix
\[
A = \begin{pmatrix}
1 & 1 & 2 & 2 \\
2 & 2 & 5 & 5 \\
0 & 0 & 3 & 3
\end{pmatrix}
\]

(a) Find a basis (which I will refer to as \( B \)) for \( \text{Nul}(A) \).

**Solution**

First row reduce \( A \):
\[
A = \begin{pmatrix}
1 & 1 & 2 & 2 \\
2 & 2 & 5 & 5 \\
0 & 0 & 3 & 3
\end{pmatrix} \sim \begin{pmatrix}
1 & 1 & 2 & 2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 3 & 3
\end{pmatrix} \sim \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
We see from this that \( \text{Nul}(A) \) has a basis

\[
\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.
\]

(b) Let \( V = \text{Nul}(A) \). Then we can define a linear transformation

\[
T: V \longrightarrow \mathbb{R}^3
\]

by \( T(v) = Av \). Write down the matrix for \( T \) in terms of the basis \( \mathcal{B} \) of \( V \) and the standard basis \( \mathcal{E} = \{ e_1, e_2, e_3 \} \) of \( \mathbb{R}^3 \).

**Solution**

The matrix we want will be \( M = [T(b_1), T(b_2)] \). Since both \( b_1 \) and \( b_2 \) are in \( \text{Nul}(A) \), \( T(b_1) = T(b_2) = \mathbf{0} \), so the matrix is

\[
M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]