Let

\[ A = \begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix}. \]

Find a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( A = PDP^{-1} \). Briefly explain yourself.

**Solution**

First we need to find the eigenvalues. For this, we look at

\[
\det(A - \lambda I) = \det \begin{pmatrix} -2 - \lambda & 12 \\ -1 & 5 - \lambda \end{pmatrix} = (\lambda - 2)(\lambda - 1),
\]

which means that the eigenvalues are \( \lambda = 1 \) and \( \lambda = 2 \). Since they are different, we know that there are two linearly independent eigenvectors for \( A \); these will form a basis for \( \mathbb{R}^2 \), so \( A \) is diagonalizable. It remains to find the eigenvectors. Let’s do \( \lambda = 1 \) first. We need to solve \((A - I)x = 0\), so we row reduce

\[
A - I = \begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix}.
\]

Now we repeat the process for \( \lambda = 2 \). Row reduce

\[
A - I = \begin{pmatrix} -4 & 12 \\ -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}.
\]

We conclude that we have eigenvectors

\[
b_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{for eigenvalue 1} \quad \text{and} \quad b_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{for eigenvalue 2}.
\]

The upshot of all of this is that

\[
A = \begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}^{-1}.
\]
(a) Write down a matrix that is not diagonalizable. Explain.

**Solution**

The matrix

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

is not diagonalizable. This is because the only eigenvalue of \( A \) is 0, but

\[ A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0, \]

so the \( \mathbb{R}^2 \) is not spanned by vectors killed by \( A \).

(b) Explain why the null space of a linear transformation is a subspace.

**Solution**

Recall that \( \text{Null}(T) = \{ x \mid T(x) = 0 \} \). So if \( x \) and \( y \) are both in \( \text{Null}(T) \) and \( c \in \mathbb{R} \), then

\[ T(x + y) = T(x) + T(y) = 0 + 0 = 0 \quad \text{and} \quad T(cx) = cT(x) = c0 = 0, \]

so \( x + y \) and \( cx \) are both in \( \text{Null}(T) \). This shows that \( \text{Null}(T) \) is a subspace.

(c) True or False: If an \( n \times n \) matrix \( A \) is diagonalizable, then every vector \( x \) in \( \mathbb{R}^n \) is an eigenvector for \( A \). Explain.

**Solution**

FALSE! For example, if

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

then

\[ Ax = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \]

which is not a multiple of \( x \), so \( x \) is not an eigenvector. But \( A \) is obviously a diagonal(izable) matrix!
(d) If the null space of a $7 \times 11$ matrix is 5-dimensional, what is the dimension of the column space?

**Solution**

The dimension of the null space is the number of nonpivot columns of $A$ and the dimension of the column space is the number of pivot columns of $A$. So if there are 5 nonpivot columns, there must be 2 pivot columns, which means that the column space is 2-dimensional.

3 Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

(a) Find a basis for $\text{Nul}(A)$.

**Solution**

To find a basis for $\text{Nul}(A)$, we simply need to solve the equation $Ax = 0$ by row reduction:

$$\begin{pmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 9 \\ 0 & -2 & -4 & -11 \\ 0 & -1 & -2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This shows that the vector

$$b = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

spans $\text{Nul}(A)$, and hence $\{b\}$ is a basis for $\text{Nul}(A)$.

(b) Find a basis for $\text{Col}(A)$.

**Solution**
As we saw in class, the pivot columns of \( A \) form a basis for \( \text{Col}(A) \).
This means that the vectors
\[
\begin{pmatrix}
2 \\
1 \\
3 \\
4 \\
2 \\
7 \\
4
\end{pmatrix},
\begin{pmatrix}
1 \\
4 \\
9 \\
7 \\
4
\end{pmatrix}
\]
are a basis for \( \text{Col}(A) \).

4 Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear transformation with the property that \( T(T(x)) = T(x) \) for every vector \( x \in \mathbb{R}^n \) (such a linear transformation is called idempotent).

(a) Write \( V \) for the image (or range) of \( T \). In other words,
\[
V = \{ T(x) \mid x \in \mathbb{R}^n \}.
\]
If \( v \in V \), then what is \( T(v) \)?
**Solution**

If \( v \in V \), then \( v = T(x) \) for some \( x \). This means that
\[
T(v) = T(T(x)) = T(x) = v.
\]
In other words, \( v \) is an eigenvector for \( T \) with eigenvalue 1.

(b) If \( x \in \mathbb{R}^n \), then what is \( T(x - T(x)) \)?
**Solution**

Again, we just need to calculate
\[
T(x - T(x)) = T(x) - T(T(x)) = T(x) - T(x) = 0.
\]
In other words, \( v \) is an eigenvector for \( T \) with eigenvalue 0.
(c) Let \( \mathcal{C} = \{c_1, c_2, \ldots, c_k\} \) be a basis for \( V \). Then we can add some more vectors, \( b_1, b_2, \ldots, b_l \) to get a basis \( \mathcal{B} \) for all of \( \mathbb{R}^n \). Show that if you replace \( b_1 \) with \( a_1 = b_1 - T(b_1) \) then you still have a basis.

**Solution**

Since the new collection has the same number of vectors as the given basis, we just need to show that the new collection spans \( \mathbb{R}^n \). Since \( T(b_1) \in V \), we can write \( T(b_1) \) as a linear combination of the \( c \)'s, which shows that

\[
b_1 \in \text{span}\{c_1, c_2, \ldots, c_k, b_1 - T(b_1)\}.
\]

So whatever the span of the new set of vectors is, it contains all the \( c \)'s and all the \( b \)'s. This means that the span of the new set of vectors is all of \( \mathbb{R}^n \).

(d) In the same way, we can replace each \( b_i \) with \( a_i = b_i - T(b_i) \). What is the matrix of \( T \) with respect to the basis \( \{c_1, c_2, \ldots, c_k, a_1, a_2, \ldots, a_l\} \)? (This is an easy question – not much work is needed!)

**Solution**

As we saw in part (a), each \( c_i \) is an eigenvector with eigenvalue 1; as we saw in part (b), each \( a_i \) is an eigenvector with eigenvalue 0. This means that \( T \) is diagonalizable, and the matrix is

\[
D = \begin{pmatrix}
1 & 0 & \cdots & \ldots & 0 \\
0 & 1 & \cdots & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \ldots & 0
\end{pmatrix}
\]

with 1’s down the diagonal until the \( k \)th place and 0’s afterwards.