Math 23 Review

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First Order Equations 1/3

- General form:

\[
\frac{dy}{dx} = f(x, y)
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\[ f(x, y) = H(x)G(y) \implies \text{Separation of variables} \]
First Order Equations 1/3

- General form:

\[ \frac{dy}{dx} = f(x, y) \]

\[ f(x, y) = H(x)G(y) \implies \text{Separation of variables} \]

- Separate variables: move x’s to one side, y’s to the other

\[ \frac{dy}{G(y)} = \frac{dx}{H(x)} \]

- Integrate both sides and solve for \( y(x) \).
Integrating factors for first order linear equations

- For an equation of the form:
  \[ y'(t) + p(t)y(t) = g(t) \]

- Form integrating factor:
  \[ \mu(t) = e^{\int p(t) \, dt} \]

- Solution:
  \[ y(t) = \frac{1}{\mu(t)} \int \mu(t) g(t) \, dt \]
Exact equations
Form:

\[ M(x, y) \, dx + N(x, y) \, dy = 0 \]

- If \( M_y = N_x \) then we can find solution \( \psi(x, y) = 0 \) via integration:
  - \( \psi_x = M, \, \psi_y = N \)

- Integrate both and form the most general \( \psi(x, y) \).
General case:

\[ p(x)y''(t) + q(x)y'(t) + r(x)y(t) = 0 \]
Second Order Equations: Linear homogeneous equations

General case:

\[ p(x)y''(t) + q(x)y'(t) + r(x)y(t) = 0 \]

General facts:

- **Superposition:** If \( y_1(t) \) and \( y_2(t) \) are linearly independent solutions then the general solution is

\[ C_1y_1(t) + C_2y_2(t) \]

- **Initial value problem:** If we are given two initial conditions and a fundamental set of solutions \( \{y_1(t), y_2(t)\} \) then we can solve the initial value problem.
Constant coefficient case

\[ ay'' + by' + cy = 0 \]

Guess the solution is of the form \( y(t) = e^{rt} \)
Constant coefficient case

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- Guess the solution is of the form \( y(t) = e^{rt} \)
- Auxillary equation: \( ar^2 + br + c = 0 \)
Constant coefficient case

\[ ay'' + by' + cy = 0 \]

- Guess the solution is of the form \( y(t) = e^{rt} \)
- Auxillary equation: \( ar^2 + br + c = 0 \)
- Roots:
  \[ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
Classification according to roots

If \( b^2 - 4ac > 0 \) then the solution is of the form

\[
y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}
\]
Classification according to roots

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y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}
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- If \( b^2 - 4ac = 0 \) then the solution is of the form
  \[
y(t) = C_1 e^{rt} + C_2 te^{rt}
  \]
Classification according to roots

- If $b^2 - 4ac > 0$ then the solution is of the form
  $$y(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

- If $b^2 - 4ac = 0$ then the solution is of the form
  $$y(t) = C_1e^{rt} + C_2te^{rt}$$

- If $b^2 - 4ac < 0$ then $r = a \pm ib$ and the solution is of the form
  $$y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt))$$
Create an independent solution from an existing solution.

- Given a single solution $y_1(t)$, guess $y_2(t) = v(t)y_1(t)$. 

Plug in and solve for a differential equation in $v(t)$:

$$y_1 v'' + (2y_0 + py_1)v' = 0$$
Reduction of Order

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- Given a single solution $y_1(t)$, guess $y_2(t) = v(t)y_1(t)$.
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  $$y_1 v'' + (2y_1' + py_1)v' = 0$$

- Solve this separable equation for $v(t)$
Inhomogeneous Equations

Example: forcing terms in spring motion
Inhomogeneous Equations

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General form:

\[ p(t)y'' + q(t)y' + r(t)y = g(t) \]
Inhomogeneous Equations

- Example: forcing terms in spring motion
- General form:

\[ p(t)y'' + q(t)y' + r(t)y = g(t) \]

- If \( y_p(t) \) is a particular solution and \( \{y_1(t), y_2(t)\} \) is a fundamental set of solutions for the homogeneous equation then the general solution is

\[ y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t) \]
Finding a particular solution

- Method of undetermined coefficients
Finding a particular solution

- Method of undetermined coefficients
- Variation of parameter
Guess a solution of a form similar to the inhomogeneous term \( g(t) \). Solve for the variable coefficients.
Undetermined Coefficients

- Guess a solution of a form similar to the inhomogeneous term $g(t)$. Solve for the variable coefficients.
- If $g(t)$ is a polynomial of degree $n$ then guess $y_p(t) = A_0 + A_1 t + \cdots + A_n t^n$. 
Undetermined Coefficients

- Guess a solution of a form similar to the inhomogeneous term $g(t)$. Solve for the variable coefficients.

- If $g(t)$ is a polynomial of degree $n$ then guess 
  $y_p(t) = A_0 + A_1 t + \cdots + A_n t^n$.

- If $g(t)$ is an exponential, guess a polynomial times an exponential.
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If $g(t)$ is an exponential, guess a polynomial times an exponential.

If $g(t)$ is a trigonometric function, guess a trig function.
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$$y_p(t) = A_0 + A_1 t + \cdots + A_n t^n.$$ If $g(t)$ is an exponential, guess a polynomial times an exponential.

If $g(t)$ is a trigonometric function, guess a trig function.

See table on page 175.
Variation of Parameters

Given an equation

\[ y'' + p(t)y' + q(t)y = g(t) \]

- If \( y_1, y_2 \) are solutions to the homogeneous equation, then guess \( y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \).
Variation of Parameters

Given an equation

\[ y'' + p(t)y' + q(t)y = g(t) \]

- If \( y_1, y_2 \) are solutions to the homogeneous equation, then guess \( y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \).
- Assume \( u'_1y_1 + u'_2y_2 = 0 \)
Variation of Parameters

Given an equation

\[ y'' + p(t)y' + q(t)y = g(t) \]

- If \( y_1, y_2 \) are solutions to the homogeneous equation, then guess \( y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \).
- Assume \( u_1' y_1 + u_2' y_2 = 0 \)
- Plugging into the equation and simplifying yields a differential equation:

\[ u_1' y_1' + u_2' y_2' = g(t) \]
Solving these two equations yields

\[ y_p(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W_r(y_1, y_2)(t)} \, dt + y_2(t) \int \frac{y_1(t)g(t)}{W_r(y_1, y_2)(t)} \, dt \]
Series solutions to second order equations

General equation:

\[ P(x)y'' + Q(x)y' + R(x)y = 0 \]

If \( x_0 \) is an ordinary point of the equation, then we can find a solution of the form

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \]
Series solutions

Method of solution:

- Plug series and power series representations for $P, Q, R$ into the equation
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- Set each coefficient to zero to find relations between the $\{a_n\}$
Series solutions

Method of solution:

- Plug series and power series representations for $P, Q, R$ into the equation
- Simplify and reindex, writing the equation as a single series
- Set each coefficient to zero to find relations between the $\{a_n\}$
- Using the recurrence relation, find the general series form of the solution.
First order linear systems

General Form:

$$\vec{x}' = A\vec{x}$$

where $A$ is a matrix.

Solutions depend on eigenvectors and eigenvalues of the matrix $A$.
Find eigenvalues of $A$:
Solve: $\det(A - \lambda I) = 0$ for all values of $\lambda$
Systems: Method of solution

- Find eigenvalues of $A$:
  Solve: $\det(A - \lambda I) = 0$ for all values of $\lambda$

- For each eigenvalue $\lambda$ find its associated eigenvector
  - i.e. find $\vec{\xi}$ so that

\[
(A - \lambda I)\vec{\xi} = 0
\]
Systems: Method of solution

- Find eigenvalues of $A$:
  Solve: $\det(A - \lambda I) = 0$ for all values of $\lambda$

- For each eigenvalue $\lambda$ find its associated eigenvector
  - i.e. find $\vec{\xi}$ so that
  \[
  (A - \lambda I)\vec{\xi} = 0
  \]

- If there are no repeated eigenvalues, the solution is then of the form:
  \[
  \vec{x} = C_1 \xi_1 e^{\lambda_1 t} + \cdots + C_n \xi_n e^{\lambda_n t}
  \]
Repeated Eigenvalues

- In some cases, we still get a full set of eigenvectors (i.e. repeated eigenvalues have multiple eigenvectors)
Repeated Eigenvalues

- In some cases, we still get a full set of eigenvectors (i.e. repeated eigenvalues have multiple eigenvectors)
- If not, we solve the *augmented eigenvector equation* for $\vec{\eta}$:

$$(A - \lambda I)\vec{\eta} = \vec{\xi}$$

where $\lambda$ is the repeated eigenvalue and $\vec{\xi}$ is an eigenvector for $\lambda$. 
Repeated Eigenvalues

- In some cases, we still get a full set of eigenvectors (i.e. repeated eigenvalues have multiple eigenvectors)

- If not, we solve the augmented eigenvector equation for $\vec{\eta}$:

$$ (A - \lambda I)\vec{\eta} = \vec{\xi} $$

where $\lambda$ is the repeated eigenvalue and $\vec{\xi}$ is an eigenvector for $\lambda$.

- Then, the piece of the general solution associated to $\lambda$ is

$$ \vec{x}(t) = \vec{\eta}te^{\lambda t} + \vec{\xi}e^{\lambda t} $$
Qualitative analysis of solutions

If $A$ is a 2x2 matrix then we can draw a phase portrait of the system and analyze the behavior of solutions qualitatively. The portraits can be sketched and classified using the eigenvalue and eigenvector data from $A$. 
Classification of 2x2 systems

If the eigenvalues are $\lambda_1, \lambda_2$ with eigenvectors $\xi_1, \xi_w$, then

- Real distinct eigenvalues
Classification of 2x2 systems

If the eigenvalues are $\lambda_1, \lambda_2$ with eigenvectors $\xi_1, \xi_w$, then

- Real distinct eigenvalues
- Complex eigenvalues
Classification of 2x2 systems

If the eigenvalues are $\lambda_1, \lambda_2$ with eigenvectors $\xi_1, \xi_w$, then

- Real distinct eigenvalues
- Complex eigenvalues
- Repeated eigenvalues
Real distinct eigenvalues

- **Same sign:**

  0 < 1 < 2: the line along 2 is a node, paths move away from the node - unstable.

  0 > 1 > 2: the line along 2 is a node, paths move towards from the node - stable (figure 9.1.1 (a)).

Opposite sign: saddle - unstable, see figure 9.1.2 (a).
Real distinct eigenvalues

- Same sign:
- $0 < \lambda_1 < \lambda_2$: the line along $\xi_2$ is a node, paths move away from the node - unstable.
Real distinct eigenvalues

- Same sign:
  - $0 < \lambda_1 < \lambda_2$: the line along $\xi_2$ is a node, paths move away from the node - unstable.
  - $0 > \lambda_1 > \lambda_2$: the line along $\xi_2$ is a node, paths move towards from the node - stable (figure 9.1.1 (a)).
Real distinct eigenvalues

- Same sign:
  - $0 < \lambda_1 < \lambda_2$: the line along $\xi_2$ is a node, paths move away from the node - unstable.
  - $0 > \lambda_1 > \lambda_2$: the line along $\xi_2$ is a node, paths move towards from the node - stable (figure 9.1.1 (a)).

- Opposite sign: saddle - unstable, see figure 9.1.2 (a)
Complex eigenvalues

\[ \lambda = a \pm ib \]

- For \( a \neq 0 \), these are spiral points
Complex eigenvalues

$$\lambda = a \pm ib$$

- For $a \neq 0$, these are spiral points
- $a > 0$ - unstable
Complex eigenvalues

\[ \lambda = a \pm ib \]

- For \( a \neq 0 \), these are spiral points
  - \( a > 0 \) - unstable
  - \( a < 0 \) - stable
Complex eigenvalues

\[ \lambda = a \pm ib \]

- For \( a \neq 0 \), these are spiral points
  - \( a > 0 \) - unstable
  - \( a < 0 \) - stable
- \( a = 0 \) - stable center
Repeated eigenvalue

\[ \lambda_1 = \lambda_2 \]

- \( \lambda_1 > 0 \): proper or improper unstable node (e.g. figure 9.1.3 (a))
Repeated eigenvalue

\[ \lambda_1 = \lambda_2 \]

- \( \lambda_1 > 0 \): proper or improper unstable node (e.g. figure 9.1.3 (a))
- \( \lambda_1 < 0 \): proper or improper stable node (e.g. figure 9.1.4 (a))
Almost linear systems

Systems of the form

$$\dot{x} = Ax + g(x)$$

can be qualitatively classified in terms of the eigenvalues/eigenvectors of $A$. See table 9.3.1
Partial Differential Equations

Three fundamental equations:

- The heat equation:

\[ \alpha^2 u_{xx} = u_t \]
Three fundamental equations:

- The heat equation:
  \[ \alpha^2 u_{xx} = u_t \]

- The wave equation:
  \[ \alpha^2 u_{xx} = u_{tt} \]
Partial Differential Equations

Three fundamental equations:

- The heat equation:
  \[ \alpha^2 u_{xx} = u_t \]

- The wave equation:
  \[ \alpha^2 u_{xx} = u_{tt} \]

- Laplace’s equation:
  \[ u_{xx} + u_{yy} = 0 \]
Separation of variables

Assume \( u(x, t) = X(x)T(t) \)
Separation of variables

- Assume \( u(x, t) = X(x)T(t) \)
- Plug in and separate variables
Separation of variables

- Assume $u(x, t) = X(x)T(t)$
- Plug in and separate variables
- Set both sides equal to a constant, yielding two ODEs
Separation of variables

- Assume $u(x, t) = X(x)T(t)$
- Plug in and separate variables
- Set both sides equal to a constant, yielding two ODEs
- Impose boundary conditions to form a two point boundary value problem
Two point boundary value problems and Fourier Series

For example:

\[ X'' + \lambda X = 0 \]

\[ X(0) = 0, \quad X(L) = 0 \]

- Find eigenvalues and eigenfunctions
Two point boundary value problems and Fourier Series

For example:

\[ X'' + \lambda X = 0 \]

\[ X(0) = 0, \quad X(L) = 0 \]

- Find eigenvalues and eigenfunctions
- Superimpose all solutions to form a Fourier series solution
Fourier Coefficients

Calculate Fourier coefficients using integral formulae:

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) \, dx \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) \, dx \]
**Fourier Coefficients**

- Calculate Fourier coefficients using integral formulae:

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) \, dx
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b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) \, dx
\]

- Remember tricks for Fourier \text{sin} and \text{cos} series