23. Verify that the transformation rules

(a) \( \mathcal{F}(x \cdot T) = \left( \frac{-1}{2\pi} \right) (\mathcal{F}T)' \); \( \mathcal{F}(x^n \cdot T) = \left( \frac{-1}{2\pi} \right)^n (\mathcal{F}T)^{(n)} \) and

(b) \( \mathcal{F}(e^{2\pi ix} \cdot T) = \mathcal{F}(s - a) \)

hold for any distribution \( T \).

24. Find \( \mathcal{F} (\cos 2\pi x) \) by expressing \( \cos 2\pi x \) in terms of complex exponentials and using (b) from the previous problem. (The answer is on page A-5.)

25. Verify that

\( (\delta(7x))' = 7\delta'(7x) \)

by

(a) applying both sides to a test function, and

(b) showing that both sides have the same Fourier transform. You may assume that all the transform rules on page A-13 hold for distributions, as well as for functions.

26. If \( \alpha \) is a function and \( T \) is a distribution, use the definition

\( \langle \alpha \ast T, \varphi \rangle = \langle T, \tilde{\alpha} \ast \varphi \rangle \)

of their convolution to show that \( \mathcal{F}(\alpha \ast T) = \mathcal{F} \alpha \cdot \mathcal{F} T \). Hint: Start with \( \langle \mathcal{F}(\alpha \ast T), \varphi \rangle \), move all the operations to \( \varphi \), then write \( \tilde{\alpha} \) as \( \mathcal{F} \mathcal{F}^{-1} \tilde{\alpha} \) and use the fact that \( \mathcal{F} \) takes products to convolutions.

27. Express the functions

\( \delta(x - a) \ast \varphi \)

and

\( (\delta(x - a)) + \delta \ast \varphi, \)

where \( \varphi \) is a test function and \( a \) is a constant, in as simple a form as possible. Your answers should not contain any distributions.

28. Find a solution to the DE

\( y'' + 8y' + 25y = f(t) \).

Express your answer in terms of a convolution of functions and also as an integral.

29. Compute \( \text{Var}(g) = \int_{-\infty}^{\infty} |x|^2 |g(x)|^2 \, dx \) for the Gaussian \( g(x) = \sqrt{\frac{a}{\pi}} e^{-ax^2} \). Also compute \( \text{Var}(g) \text{Var}(\mathcal{F}g) \). Could the constant \( \frac{1}{16\pi^2} \) in Heisenberg’s inequality be any larger?

30. The Gabor transform of \( f \) is defined by

\[ \mathcal{G}f(s, m) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-2\pi isx} e^{-(x-m)^2/2} \, dx. \]

Show that

\[ f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{G}f(s, m) e^{2\pi isx} e^{-(x-m)^2/2} \, ds \, dm. \]
Hint: The expression just above for $f(x)$ is a triple integral. Recognize the innermost integral with respect to $u$ (assuming the name of the variable of integration in the definition of $Gf$ has been changed from $x$ to $u$) as a Fourier transform, and the integral with respect to $s$ as an inverse Fourier transform.

31. (a) From class or page 420, the two distributions $\mathcal{F}(e^{i\pi x^2})$ and $e^{-i\pi s^2}$ satisfy

$$\mathcal{F}(e^{i\pi x^2}) = ce^{-i\pi s^2}$$

for some constant $c$. The value of $c$ can be determined by applying these distributions to a test function—a convenient one is $\phi(x) = e^{-\pi x^2}$. Find $c$ by first expressing both sides of the equation

$$\langle \mathcal{F}(e^{i\pi x^2}), e^{-i\pi s^2} \rangle = \langle e^{i\pi x^2}, \mathcal{F}(e^{-i\pi s^2}) \rangle$$

as an integral. This gives the second line of Exercise 7.45 on p. 466. Then do the rest of Exercise 7.45.

(b) What are $\mathcal{F}^{-1}(e^{-i\pi s^2})$ and $\mathcal{F}(e^{-i\pi x^2})$?

(c) Use the dilation rule on p. A-13 to find $\mathcal{F}(e^{-i\pi ax^2})$ where $a$ is any real constant. Consider the cases $a < 0$, $a = 0$ and $a > 0$ separately.

32. Find the solution $u(x, t)$ to the heat equation

$$\alpha^2 u_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0$$

which satisfies the initial condition

$$u(x, 0) = \delta(x - a), \quad -\infty < x < \infty$$

where $a$ is a constant. Simplify your answer as much as possible.

33. Assume that $f$ is a function satisfying $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$ and $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = 1$. Show that if $u$ is the solution to the free Schrödinger equation

$$u_t = \frac{i\lambda}{4\pi} u_{xx}$$

satisfying

$$u(x, 0) = f(x),$$

then the integral

$$\int_{-\infty}^{\infty} |u(x, t)|^2 \, dx = 1$$

for each value of $t > 0$. Hint: Do not work directly with $u$. Use the Parseval or Plancherel identity and work with $U = \mathcal{F}u$. 
34. (Divergence theorem review problem) Evaluate the integral
\[ \int_{\partial R} (x^2i - 2xyj) \cdot n \, ds \]
where \( R \) is the region enclosed by the ellipse \( x^2/a^2 + y^2/b^2 = 1 \) and \( n \) is the outward-pointing unit normal vector to the boundary \( \partial R \) of \( R \).

35. Find the solution \( u(x, y) \) to Laplace’s equation
\[ u_{xx} + u_{yy} = 0 \quad -\infty < x < \infty, \quad y > 0 \]
on the upper half-plane which satisfies the boundary condition
\[ u(x, 0) = H(x) = \text{Heaviside function}, \quad -\infty < x < \infty. \]
Describe the level curves of \( u \). What is the value of \( u \) along each of them? (Hint: The solution \( u \) can be expressed in a simple form in terms of the angle \( \theta = \tan^{-1} \frac{y}{x} \) of polar coordinates.)

36. Find the solution to the non-homogeneous wave equation
\[ c^2u_{xx}(x, t) + f(x, t) = u_{tt}(x, t), \quad -\infty < x < \infty, \quad t > 0 \]
satisfying the homogeneous initial conditions
\[ u(x, 0) = 0 \quad \text{and} \quad u_t(x, 0) = 0 \quad \text{for} \quad -\infty < x < \infty. \]
Write the solution in as simple a form as possible. (Hint: This is very similar in some ways to the problem
\[ \alpha^2u_{xx} + f(x, t) = u_t, \quad -\infty < x < \infty, \quad t > 0 \]
\[ u(x, 0) = 0, \quad -\infty < x < \infty \]
which we solved in class. Also, the forms of the answers to both problems are similar.)