Metric spaces are a useful concept often used to bridge the gap between geometry and topology. A metric space consists of a set \( X \) and a function \( d : X \times X \to \mathbb{R}^1 \) such that:

1. \( d(x, y) \geq 0 \) with \( d(x, y) = 0 \) if and only if \( x = y \)
2. \( d(x, y) = d(y, x) \)
3. \( d(x, y) + d(y, z) \geq d(x, z) \)

The function \( d \) is referred to as a distance function or metric on \( X \).

**Problem 1A:** [20 pts]

(a) Show that the \( \mathbb{R}^3 \) together with the function

\[
d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}
\]

is a metric space. (Try to argue geometrically rather than computationally).

A subset \( X \subset \mathbb{R}^3 \) is \( C^1 \) path-connected if for every pair of points \( x, y \in X^1 \), there is a \( C^1 \) curve \( \gamma : [a, b] \to \mathbb{R}^3 \) such that \( \gamma \) lies entirely in \( X \) and \( \gamma(a) = x, \gamma(b) = y \). In other words, every pair of points can be connected by a \( C^1 \) curve.

For such a subset \( X \), we can define a new function

\[
d_X(x, y) = \min \left\{ \int_a^b \|\dot{\gamma}(t)\| dt : \gamma \text{ is a } C^1 \text{ curve into } X \text{ with } \gamma(a) = x, \gamma(b) = y \right\}
\]

(b) Show that \( (X, d_X) \) is a metric space.

When \( X = \{x_1^2 + x_2^2 + x_3^2 = 1\} \), i.e., the unit sphere centered at \((0, 0, 0)\), we shall see later that the length minimizing curves are the great circles (intersections of the sphere with planes passing through the origin)

(c) Use the fact that \( d_X \) is a metric space to prove the following result:

"The sum of any two angles formed at the vertex of a triangular pyramid is greater than the third angle."\(^7\)

(Hint: think about lengths of arcs of circles in terms of angles.)

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\(^7\)The Cartesan product of sets \( X \times Y \) consists of all pairs \((x, y)\) such that \( x \in X \) and \( y \in Y \).

\(^1\)The notation \( x \in X \) mean \( x \) is an element of the set \( X \).
**Def:** Given a point \( p \) contained in a subset \( X \) of \( \mathbb{R}^2 \) (or \( \mathbb{R}^3 \)), the *tangent space to \( X \) at \( p \)*, denoted \( T_pX \), is the collection of all pairs \( \{(p, v)\} \) such that there exists a \( C^1 \) curve \( \gamma: (-\epsilon, \epsilon) \to \mathbb{R}^2 \) (or \( \mathbb{R}^3 \)) with its image entirely in \( X \) such that

- \( \gamma(0) = p \)
- \( \gamma'(0) = v \).

In other words \( (p, v) \) is in \( T_pX \) if there is a \( C^1 \) curve in \( X \) passing through \( p \) with tangent vector \( v \) at \( p \). Note: with this definition, the curve must lie inside \( X \) defined on both sides of \( p \).

The tangent space is defined in this slightly strange way to emphasize that the tangent vectors start at \( p \), whereas the vectors \( v \) alone just represent a magnitude and direction and do not encode a position in space.

To help visualization, we often think about the *embedded tangent space*, denoted \( E_pX \) which consists of all points \( \{p + v\} \) where \( (p, v) \in T_pX \), i.e. the endpoints of all vectors \( v \) starting at \( p \).

**Problem 1B:** [20 pts]

(a) Show that if \( (p, v) \in T_pX \) and \( \lambda \in \mathbb{R} \) then \( (p, \lambda v) \in T_pX \).

(Note this means that the embedded tangent space is the union of \( \{p\} \) with lines through \( p \), although there maybe zero or infinitely many.)

Describe the embedded tangent spaces of the following subsets \( X \) at the given points \( p \). Briefly justify your answers.

(b) \( X = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2, \ p = (1, 0). \)

(c) \( X = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2, \ p = (0, 0). \)

(d) \( X = \{\max |x|, |y| = 1\} \subset \mathbb{R}^2, \ p = (1, 1). \)

(e) \( X \) is the image of the “figure-eight” curve \( \gamma(t) = (\sin(2t), \sin t), \ p = (0, 0). \)

(f) \( X = \{z = x^2 + y^2\} \subset \mathbb{R}^3, \ p = (1, 0, 1). \)