Important Densities

May 2, 2006
Continuous Uniform Density

- Let $U$ be the random variable whose value represents the outcome of the experiment consisting of choosing a real number at random from the interval $[a, b]$.

\[
f(\omega) = \begin{cases} 
  \frac{1}{b - a}, & \text{if } a \leq \omega \leq b, \\
  0, & \text{otherwise}.
\end{cases}
\]
Exponential Density

- The exponential density function is defined by

\[ f(x) = \begin{cases} 
\lambda e^{-\lambda x}, & \text{if } 0 \leq x < \infty, \\
0, & \text{otherwise}. 
\end{cases} \]

- \( \lambda \) is any positive constant, depending on the experiment.
The cumulative distribution function

- Let $T$ be an exponentially distributed random variable with parameter $\lambda$.

- If $x \geq 0$, then we have

$$F(x) = P(T \leq x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}.$$
The “Memoryless” Property

\[ P(T > r + s \mid T > r) = P(T > s) \]
Gamma Density

- Define $X_1, X_2, \ldots$ to be a sequence of independent exponentially distributed random variables with parameter $\lambda$.

- Consider a time interval of length $t$.

- Let $Y$ denote the random variable which counts the number of emissions that occur in the time interval.
• Let $S_n$ denote the sum $X_1 + X_2 + \cdots + X_n$
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$$P(Y = n) = P(S_n \leq t \text{ and } S_{n+1} > t) .$$
Exponential Density ...

- Let $S_n$ denote the sum $X_1 + X_2 + \cdots + X_n$

  \[ P(Y = n) = P(S_n \leq t \text{ and } S_{n+1} > t) \, . \]

  \[ = P(S_n \leq t) - P(S_{n+1} \leq t) \, . \]
The density of $S_n$ is called the gamma density with parameters $\lambda$ and $n$:

$$g_n(x) = \begin{cases} 
\lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x}, & \text{if } x > 0, \\
0, & \text{otherwise.}
\end{cases}$$

The cumulative distribution function is

$$G_n(x) = \begin{cases} 
1 - e^{-\lambda x} \left( 1 + \frac{\lambda x}{1!} + \cdots + \frac{(\lambda x)^{n-1}}{(n-1)!} \right), & \text{if } x > 0, \\
0, & \text{otherwise.}
\end{cases}$$
• Then

\[ P(Y = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \]
Example

• Suppose that customers arrive at random times at a service station with one server, and suppose that each customer is served immediately if no one is ahead of him, but must wait his turn in line otherwise.

• How long should each customer expect to wait?
Assume that the interarrival times between successive customers are given by random variables $X_1$, $X_2$, $\ldots$, $X_n$ with an exponential cumulative distribution function given by

$$F_X(t) = 1 - e^{-\lambda t}.$$ 

Assume, too, that the service times for successive customers are given by random variables $Y_1$, $Y_2$, $\ldots$, $Y_n$

$$F_Y(t) = 1 - e^{-\mu t}.$$

Important Densities
Functions of a Random Variable

**Theorem.** Let $X$ be a continuous random variable, and suppose that $\phi(x)$ is a strictly increasing function on the range of $X$. Define $Y = \phi(X)$. Suppose that $X$ and $Y$ have cumulative distribution functions $F_X$ and $F_Y$ respectively. Then these functions are related by

$$F_Y(y) = F_X(\phi^{-1}(y)).$$

If $\phi(x)$ is strictly decreasing on the range of $X$, then

$$F_Y(y) = 1 - F_X(\phi^{-1}(y)).$$
Corollary. Let $X$ be a continuous random variable, and suppose that $\phi(x)$ is a strictly increasing function on the range of $X$. Define $Y = \phi(X)$. Suppose that the density functions of $X$ and $Y$ are $f_X$ and $f_Y$, respectively. Then these functions are related by

$$f_Y(y) = f_X(\phi^{-1}(y)) \frac{d}{dy} \phi^{-1}(y).$$

If $\phi(x)$ is strictly decreasing on the range of $X$, then

$$f_Y(y) = -f_X(\phi^{-1}(y)) \frac{d}{dy} \phi^{-1}(y).$$
Simulation

Corollary. If $F(y)$ is a given cumulative distribution function that is strictly increasing when $0 < F(y) < 1$ and if $U$ is a random variable with uniform distribution on $[0, 1]$, then

$$Y = F^{-1}(U)$$

has the cumulative distribution $F(y)$. 

Important Densities
Normal Density

- The normal density function with parameters $\mu$ and $\sigma$ is defined as follows:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$
Normal Density

- The normal density function with parameters $\mu$ and $\sigma$ is defined as follows:
  \[ f_X(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \]

- The parameter $\mu$ represents the "center" of the density.

- The parameter $\sigma$ is a measure of the "spread" of the density.
The Cumulative Distribution

- The cumulative distribution function is given by the formula

\[
F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu)^2/2\sigma^2} \, du .
\]
Normal Density ...

Important Densities
The Standard Normal Random Variable

- A normal random variable with parameters \( \mu = 0 \) and \( \sigma = 1 \) is said to be a *standard* normal random variable.
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- A normal random variable with parameters $\mu = 0$ and $\sigma = 1$ is said to be a *standard* normal random variable.

- If we write

  $$X = \sigma Z + \mu ,$$

  then $X$ is a normal random variable with parameters $\mu$ and $\sigma$.

- The cumulative distribution of $X$ in terms of $Z$ is

  $$F_X(x) = F_Z \left( \frac{x - \mu}{\sigma} \right).$$