Continuous Density Functions

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Bertrand’s Paradox (cont’d)

• A chord of a circle is a line segment both of whose endpoints lie on the circle.

• Suppose that a chord is drawn at random in a unit circle.

• What is the probability that its length exceeds $\sqrt{3}$?
Spinners (revisited)

• A spinner consists of a circle of unit circumference and a pointer.

• Let’s simulate this experiment such that we produce a graph bar with the property that on each interval, the area, rather than the height, of the bar is equal to the fraction of outcomes that fell in the corresponding interval.

• Use the program *Areabargraph*.
• We would like

\[ P(c \leq X < d) = d - c. \]

• If we let \( E = [c, d] \), then we can write the above formula in the form

\[ P(E) = \int_E f(x) \, dx, \]

where \( f(x) \) is the constant function with value 1.

• The function \( f(x) \) is called the \textit{density function} of the random variable \( X \).
Sum of random numbers

- Choose two random real numbers in \([0, 1]\) and add them together.
- Let \(X\) be the sum.
- How is \(X\) distributed?
Sum of random numbers ...
Sum of random numbers ...

- It appears that the function defined by

\[ f(x) = \begin{cases} 
  x, & \text{if } 0 \leq x \leq 1, \\
  2 - x, & \text{if } 1 < x \leq 2 
\end{cases} \]

fits the data very well.
Sum of random numbers ...

- Suppose that we choose 100 random numbers in \([0, 1]\), and let \(X\) represent their sum.

- How is \(X\) distributed?
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Darts

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• Suppose we throw a dart once so that it hits the target, and we observe where it lands.
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• A game of darts involves throwing a dart at a circular target of *unit radius*.

• Suppose we throw a dart once so that it hits the target, and we observe where it lands.

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\[
P(E) = \frac{\text{area of } E}{\text{area of target}} = \frac{\text{area of } E}{\pi}.
\]
• This can be written in the form

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• if \( E = \{ (x, y) : x^2 + y^2 \leq a^2 \} \) is the event that the dart lands within distance \( a < 1 \) of the center of the target, then

\[ P(E) = \frac{\pi a^2}{\pi} = a^2 . \]
Sample Space Coordinates

• A sample space $\Omega$ which is a subset of $\mathbb{R}^n$ is called a \textit{continuous sample space}.

• Let $X$ be a random variable which represents the outcome of the experiment.

• Such a random variable is called a \textit{continuous random variable}.
Density Functions of Continuous Random Variables

• Let $X$ be a continuous real-valued random variable. A density function for $X$ is a real-valued function $f$ which satisfies

$$P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx$$

for all $a, \ b \in \mathbb{R}$.

• if $E$ is a subset of $\mathbb{R}$, then

$$P(X \in E) = \int_{E} f(x) \, dx .$$
Examples

- **The spinner**: \( \Omega = [0, 1] \) and

  \[
  f(x) = \begin{cases} 
  1, & \text{if } 0 \leq x < 1, \\
  0, & \text{otherwise}.
  \end{cases}
  \]

- **The dart game**: \( \Omega = \{(x, y) : x^2 + y^2 \leq 1\} \), and

  \[
  f(x, y) = \begin{cases} 
  1/\pi, & \text{if } x^2 + y^2 \leq 1, \\
  0, & \text{otherwise}.
  \end{cases}
  \]
Cumulative Distribution Functions of Continuous Random Variables

• Let $X$ be a continuous real-valued random variable.

• Then the *cumulative distribution* function of $X$ is defined by the equation

$$F_X(x) = P(X \leq x).$$
Theorem. Let $X$ be a continuous real-valued random variable with density function $f(x)$. Then the function defined by

$$F(x) = \int_{-\infty}^{x} f(t) \, dt$$

is the cumulative distribution function of $X$. Furthermore, we have

$$\frac{d}{dx} F(x) = f(x).$$