Each student should work on a project throughout the quarter. It is encouraged that you work together in groups of two or three students. At the end of the quarter, each group will present their project giving a short talk. Here is a list of possible topics. If you have your own topic in mind you are more than welcome to discuss it with me.

1. **Domino tilings of Aztec diamonds.** Counting the number of domino tilings of a given shape is a difficult problem in general. However, for the so-called Aztec diamond, the number of domino tilings has a very simple formula.
   - [Aig] Pages 44–50.

2. **Random walks in \( \mathbb{Z}^d \).** The probability that a random walk in \( \mathbb{Z}^d \) returns to the origin is 1 for \( d = 1, 2 \), but strictly less than 1 for \( d \geq 3 \). In other words, you shouldn’t get drunk unless you move in at most two dimensions.
   - [Aig] Pages 5–9.

3. **The Gessel-Viennot method.** This is a beautiful formula to enumerate \( n \)-tuples of nonintersecting lattice paths. The answer is given by a determinant of binomial coefficients, and the proof is based on the combinatorics of involutions.
   - [Aig] Section 5.4.
   - [EC1] Section 2.7.

4. **The descent number and the major index.** We say that \( i \) is a descent of a permutation \( \pi \in S_n \) if \( \pi_i > \pi_{i+1} \). The major index of \( \pi \), denoted \( \text{maj}(\pi) \), is defined as the sum of all descents in \( \pi \). For example, \( \text{maj}(12\cdots n) = 0 \) and \( \text{maj}(n \cdots 21) = 1+2+\cdots+(n-1) \). On the other hand, an inversion of \( \pi \) is a pair \((i,j)\) such that \( i < j \) and \( \pi_i > \pi_j \). The inversion number of \( \pi \), denoted \( \text{inv}(\pi) \), is the number of inversions of \( \pi \).

The goal of this project is to show that for any value \( k \), the number of permutations \( \pi \in S_n \) with \( \text{maj}(\pi) = k \) is the same as the number of permutations \( \pi \in S_n \) with \( \text{inv}(\pi) = k \). In other words, the major index \( \text{maj} \) is equidistributed with the number of inversions \( \text{inv} \), that is,

\[
\sum_{\pi \in S_n} q^{\text{maj}(\pi)} = \sum_{\pi \in S_n} q^{\text{inv}(\pi)}.
\]

A stronger version of this is the fact that the joint distribution of \( \text{maj} \) and \( \text{inv} \) is symmetric, that is,

\[
\sum_{\pi \in S_n} q^{\text{maj}(\pi)} t^{\text{inv}(\pi)} = \sum_{\pi \in S_n} q^{\text{inv}(\pi)} t^{\text{maj}(\pi)}.
\]

5. **Combinatorial proofs of the Lagrange Inversion Formula.** In class we will see an analytic proof of the Lagrange Inversion Formula. However, it is possible to give a combinatorial proof, interpreting the coefficients as counting plane forests.
   - [EC2] Section 5.4.

6. **Walks in graphs.** The number of walks of given length between two vertices of a graph can be expressed in terms of the eigenvalues of its adjacency matrix. This is a nice connection of combinatorics and linear algebra.
   - [St] Section 1.

7. **The transfer-matrix method.** An application of counting walks in graphs to other problems in enumerative combinatorics.
   - [EC1] Section 4.7.

8. **Viennot’s geometric construction of the RSK correspondence.** In class we will discuss the RSK algorithm, which gives a correspondence between permutations and pairs of standard Young tableaux. A beautiful geometric description of this correspondence is due to Viennot.

9. **Increasing and decreasing subsequences of permutations.** This theory is an application of the Robinson-Schensted correspondence (or RSK algorithm).
   - [BS] Section 5.

10. **A combinatorial proof of the unimodality of the Gaussian polynomials.** In class we discussed an algebraic proof that the $q$-binomial coefficients are unimodal. A direct combinatorial proof was given by Kathy O’Hara.

11. Read a paper from a combinatorics journal and present it in class. You are encouraged to talk with me to pick a suitable paper. Here are some interesting journals that you can find in the library or online.
    - *Journal of Combinatorial Theory A*,
    - *Electronic Journal of Combinatorics*,
    - *Journal of Algebraic Combinatorics*,
    - *European Journal of Combinatorics*,
    - *Annals of Combinatorics*,
    - *Discrete Mathematics*. 