Exercises for Week 4

Solutions to these problems are due on Wednesday, October 21.

1. Compute
\[ \mu(13/2/45/68/7, 13457/268) \]
in the set partition lattice \( \Pi_s \).

2. Let \( P \) be a poset with a minimum element \( \hat{0} \), and suppose that the element \( x \in P \) covers only one other element, \( y \). Prove that if \( y \neq \hat{0} \), then \( \mu(\hat{0}, x) = 0 \).

3. Define the function \( f(n) \) by
\[ \sum_{d|n} f(d) = \log n. \]
Prove that
\[ f(n) = \begin{cases} \log p & \text{if } n \text{ is a power of the prime } p, \\ 0 & \text{otherwise} \end{cases} \]

Exercises 4–6 concern the symmetric chain decomposition for \( (2^{[n]}, \subseteq) \) constructed by Greene and Kleitman. Recall that to find the successor of \( A \subseteq [n] \) in this SCD, we write out its characteristic vector, match \( 0s \) and \( 1s \) from left to right, and then take \( A \cup \{k\} \) if \( k \) is the leftmost unmatched \( 0 \), or stop if there are no unmatched \( 0s \). For example, to compute the successor of
\[ A = \{3, 4, 6, 7, 8\} \subseteq [9], \]
we have
\[ \chi(A) = 001101110, \]
so matching the \( 0s \) and \( 1s \), we have
\[ 0 \ 01 \ 101110. \]
Thus the successor of \( A \) is \( \sigma(A) = A \cup \{9\} \). Given a set \( A \), let \( U_0(A) \) denote the elements of \([n]\) which correspond to unmatched \( 0s \), and \( U_1(A) \) denote the elements of \([n]\) which correspond to unmatched \( 1s \).

4. Prove that for any \( A \subseteq [n] \), the rightmost unmatched \( 1 \) lies to the left of the leftmost unmatched \( 0 \).

5. Take \( A \subseteq [n] \), and suppose that
\[ U_1(A) = \{i_1, i_2, \ldots, i_t\}, \]
\[ U_0(A) = \{i_{j+1}, i_{j+2}, \ldots, i_t\}, \]
where \( i_1 < i_2 < \cdots < i_t \). Prove that if \( U_0(A) \neq \emptyset \) then
\[ U_1(\sigma(A)) = \{i_1, i_2, \ldots, i_{j+1}\}, \]
\[ U_0(\sigma(A)) = \{i_{j+2}, i_{j+3}, \ldots, i_t\}. \]

6. Prove that this construction gives an SCD of \( (2^{[n]}, \subseteq) \) by showing that if \( A \subseteq [n] \) with \( U_1(A) = \emptyset \) then
\[ C : A \subseteq \sigma(A) \subseteq \sigma^2(A) \subseteq \cdots \subseteq \sigma^{U_0(A)}(A) \]
is a symmetric chain beginning at \( A \) and ending at \( A \cup U_0(A) \).