Scheduling games

10 football teams
7 Saturdays
each team must play every other.

Solution:

and repeat...

Terminology

A perfect matching for the graph $G$ is a collection of edges so that every vertex is in precisely one edge.

These are also called 1-factors.

A 1-factorization of $G$ is a partition of its edges into 1-factors.

Aside: How many 1-factorizations does $K_{14}$ have?

$98, 758, 655, 816, 833, 727,$

$741, 339, 583, 040.$

(1 million times #stars in universe)

Our construction generalizes to show that $K_{2n}$ has a 1-factorization for all $n$.

What about generalizing this?

$K_{2n} = \{ \text{all 2-subsets of } [2n] \}.$

Suppose $k|n$. Can we partition the $k$-subsets of $[n]$ nicely?

A parallel class is a set of $\frac{n}{k}$ disjoint elements of $\binom{[n]}{k}$, the set of all $k$-subsets of $[n]$.

If $k|n$, can we partition $\binom{[n]}{k}$ into parallel classes?

If we could, how many parallel classes would we need?

Answer: $\frac{n}{k} = \binom{n-1}{k-1}$. 
Baranyai’s Theorem (1973):
If $k|n$, then $(\binom{n}{k})$ can be partitioned into $(\binom{n-1}{k-1})$ parallel classes.

Proof: We give the proof due to Brouwer and Schrijver. This proof is inductive, but we need a "catalytic" variable to make the induction work.

Inductive claim: Suppose $k|n$. For any $0 \leq \lambda \leq n$, there exists a collection $a_1, a_2, \ldots, a_{\binom{n-1}{k-1}}$ of $\frac{n}{k}$-partitions of $[n]$ with the property that each subset $S \subseteq [n]$ occurs in precisely $\binom{n-\lambda}{k-1}$ of the $\frac{n}{k}$-partitions $a_i$. Note: $\lambda = 0$ is trivial... $a_i = \binom{\emptyset}{\emptyset}$.

$\lambda = n$ implies the theorem.

Def: An $m$-partition of the set $X$ is a multiset $A$ of $m$ pairwise disjoint subsets of $X$ (some of which may be empty) whose union is $X$.

Suppose the claim holds for $\lambda = 0$. We construct a network.
A flow in this network:
all edges $\sigma \to a_i$ flow $\frac{k-151}{n-\lambda}$
all edges $a_i \to S$ flow $\frac{k-151}{n-\lambda}$
all edges $S \to \tau$ flow $(\frac{n-\lambda-1}{k-151})$.

Out-flow at $a_i$:
$$\sum_{S \ni a_i} \frac{k-151}{n-\lambda} = \frac{1}{n-\lambda} (mk - \sum 151)$$
$$= \frac{1}{n-\lambda} (mk - \lambda)$$
$$= 1$$
(where $m = \frac{n}{k}$.)

In-flow at $S$:
$$\sum_{i : S \ni a_i} \frac{k-151}{n-\lambda} = \frac{k-151}{n-\lambda} (n-\lambda)$$
$$= (n-\lambda-1)$$

This verifies that this is a flow.
Since all edges leaving $\sigma$ are at capacity, this is a maximum flow.

Also, the edges entering $\tau$ are saturated. Hence they must be saturated in any maximum flow.

Our proof of the Max-Flow Min-Cut theorem actually proved more—if all capacities are integers, then there is an integral maximum flow.

So this network has a maximum integral flow. What does this flow look like? For each $i$, it assigns a flow of 1 to the edge $a_i \to S$ for some member $S$ of $a_i$.

Call this member “special.”

Now the flow assigns $(\frac{n-\lambda-1}{k-151})$ to each edge $S \to \tau$, so each $S \subseteq [\lambda]$ is special for precisely this many $\frac{\lambda}{k}$-partitions $a_i$.

It is time to construct a set $B_1, B_2, \ldots, B_{\frac{\lambda}{k}+1}$ of $\frac{\lambda}{k}$-partitions of $[\lambda+1]$.

Do this by replacing the special member $S \ni a_i$ by $S \cup [\lambda+1]$ to form $B_i$. 
Each subset $T \subseteq [d+1]$ occurs
\[ \binom{n-(d+1)}{k-1} \]
times in $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{\binom{n-k}{d+1}}$,
completing the inductive step. \qed