Polya Counting (lecture 1/2)

Usually coins have two sides, heads and tails, but if you want to cheat at coin flipping, you'll want 3 coins: HT, HH, TT.

Importantly, TH is the "same" as HT, so there are 3, not 4, coins.

This is because the group $\mathbb{Z}_2$ acts on the set of coins, and its orbits are {HT, TH, {HH, and {TT}.

Polya Counting generalizes this idea to any situation where a group acts on the objects of interest.

Example: Suppose we color the four corners of a square with the colors red and blue. How many different colorings are there if we allow the square to move around?

So, for example, we consider the colorings

\[
\begin{align*}
R & \rightarrow B \\
B & \rightarrow B \\
B & \rightarrow B \\
R & \rightarrow B
\end{align*}
\]

and

\[
\begin{align*}
R & \rightarrow B \\
B & \rightarrow B \\
B & \rightarrow B \\
R & \rightarrow B
\end{align*}
\]

to be equivalent.

The first step is to figure out which group is acting on these colorings.

This group is $D_4$, the symmetries of the square.

To be precise, label the vertices as

\[
\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 4 \\
3 & \rightarrow 1 \\
4 & \rightarrow 3
\end{align*}
\]

Then rotating clockwise by 90° is

\[
\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}
\]

and flipping about the line RB is

\[
\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}
\]

These two elements generate

\[
D_4 = \{e, \rho, \rho^2, \rho^3, \tau, \tau \rho, \tau \rho^2, \tau \rho^3\}
\]

Let $X = \{all\ colorings\ of\ the\ square\}$. For all $x \in X$ and $g \in G$, $x$ is equivalent to $gx$.

Therefore, to count inequivalent colorings, we want to count orbits. The orbit of $x \in X$ is

\[
\text{orb}(x) = \{gx : g \in G\}.
\]

Orbit-Counting Lemma (aka Burnside's Lemma): Suppose the group $G$ acts on the set $X$. Then

\[
\# \text{orbits} = \frac{1}{|G|} \sum_{g \in G} \text{Fix}(g),
\]

where $\text{Fix}(g) = \{x \in X : gx = x \}$. 

To prove this, we introduce a bit more notation and prove a lemma.

Suppose $G$ acts on the set $X$.

The stabilizer, $\text{stab}(x)$ of $x \in X$ is defined as

\[ \text{stab}(x) = \{ g \in G : gx = x \} \]

Note that $\text{stab}(x)$ is a subgroup of $G$.

**Orbit-Stabilizer Theorem:**

The map $g \cdot \text{stab}(x) \mapsto gx$ is a bijection between $G/\text{stab}(x)$ and $\text{orb}(x)$.

**Proof:** First we must show the map is well-defined. Suppose

\[ g \cdot \text{stab}(x) = h \cdot \text{stab}(x) \]

so $g = hs$ for some $s \in \text{stab}(x)$. Then we have

\[ gx = hsx = hx \]

so the map is well-defined.

Clearly the map is onto: if $y \in \text{orb}(x)$, then $y = gx$ for some $g \in G$, so $y$ is the image of $g \cdot \text{stab}(x)$.

Now suppose $gx = hx$. Then $g^{-1}hx = x$, so $g^{-1}h \in \text{stab}(x)$, and thus $g \cdot \text{stab}(x) = h \cdot \text{stab}(x)$. \[ \blacksquare \]

**Proof of the Orbit-Counting Lemma:**

Consider

\[ \sum_{g \in G} |\text{fix}(g)| = \left| \left\{ (g,x) \in G \times X : gx = x \right\} \right| \]

\[ = \sum_{x \in X} |\text{stab}(x)|. \]

By the Orbit-Stabilizer Theorem,

\[ |G/\text{stab}(x)| = |\text{orb}(x)|, \]

so using Lagrange's Theorem,

\[ \frac{|G|}{|\text{stab}(x)|} = |\text{orb}(x)|, \]

and thus

\[ |\text{stab}(x)| = \frac{|G|}{|\text{orb}(x)|}. \]

**Making this substitution yields**

\[ \sum_{g \in G} |\text{fix}(g)| = \sum_{x \in X} \frac{|G|}{|\text{orb}(x)|} \]

\[ = |G| \sum_{x \in X} \frac{1}{|\text{orb}(x)|}. \]

Now note that each orbit $A$ occurs $|A|$ times in this sum, so the sum is the # of orbits:

\[ \sum_{x \in X} \frac{1}{|\text{orb}(x)|} = \sum_{A \in \text{orbits}} \frac{|A|}{|A|} \]

\[ = \sum_{A \in \text{orbits}} |A| \]

\[ = \# \text{orbits} \]

Therefore

\[ \sum_{g \in G} |\text{fix}(g)| = |G| (\# \text{orbits}). \] \[ \blacksquare \]
Returning to the colored squares example, we see that we need to count fixed "points", i.e., fixed colorings.

| $g \in G$ | $g^4/3$ | $|\text{Fix}(g)|$ |
|-----------|---------|------------------|
| $e$       | $1^4/3$ | 16               |
| $\sigma$  | $3^4/3$ | 2                |
| $\sigma^2$| $3^2/3$ | 4                |
| $\sigma^3$| $3^3/3$ | 2                |
| $\tau$    | $2^3$   | 8                |
| $\tau \rho$| $2^3 = 4^3$ | 4                |
| $\tau \rho^2$| $2^3 = 4^3$ | 8                |
| $\tau \rho^3$| $2^3 = 4^3$ | 4                |
|           |         | 48               |

Therefore, the number of inequivalent colorings is $48/4 = 6$. 