Enumeration under group action (2/2)

Recall the general set-up from last time.

We have a set \( X \) of objects, with a group \( G \) of symmetries.

The group \( G \) "acts" on \( X \).

We identify two objects of \( X \) if there is a group element that maps one to the other.

This defines an equivalence relation (why?) on \( X \) by
\[
y \sim x \text{ if } y = gx \text{ for some } g \in G.
\]

The equivalence classes are the orbits,
\[
\text{orb}(x) = \{ gx : g \in G \}.
\]

We want to count inequivalent objects, or in other words, orbits.

**Orbit-Counting Lemma:** Suppose the group \( G \) acts on the set \( X \). Then
\[
\# \text{orbits} = \text{average of fixed points} = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|,
\]
where \( \text{fix}(g) = \{ x \in X : gx = x \} \).

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Suppose we have a group \( G \) acting on a set of objects \( X \). Without loss, we may assume \( G \) is a permutation group.

**Def:** The cycle index monomial of a permutation \( \pi \) is
\[
x_{c_1}^1 x_{c_2}^2 \ldots x_{c_k}^k
\]
where \( c_k \) denotes the number of cycles in \( \pi \) of length \( k \).

Note that as long as \( X \) is finite, these will really be monomials.

**Def:** The cycle index of the group \( G \) is the average of the cycle indices of all its elements. We denote this by \( Z(G) \).

**Example:** \( G = D_4 \)

<table>
<thead>
<tr>
<th>( g \in G )</th>
<th>( 1 \quad 2 \quad 3 )</th>
<th>cycle index</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>1, 2, 3</td>
<td>( x_1^4 )</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>1, 1</td>
<td>( x_1^4 )</td>
</tr>
<tr>
<td>( \sigma^2 )</td>
<td>1, 3</td>
<td>( x_1^4 )</td>
</tr>
<tr>
<td>( \sigma^{-1} )</td>
<td>1, 3</td>
<td>( x_1^4 )</td>
</tr>
<tr>
<td>( \tau )</td>
<td>1, 3</td>
<td>( x_1^4 x_2 )</td>
</tr>
<tr>
<td>( \tau \sigma )</td>
<td>1, 3</td>
<td>( x_1^4 x_2 )</td>
</tr>
<tr>
<td>( \tau \sigma^2 )</td>
<td>1, 3</td>
<td>( x_1^4 x_2 )</td>
</tr>
<tr>
<td>( \tau \sigma^3 )</td>
<td>1, 3</td>
<td>( x_1^4 x_2 )</td>
</tr>
</tbody>
</table>

So, the cycle index of \( D_4 \) is:
\[
\frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4).
\]
How can we get the number of 2-colored squares out of \( \mathbb{Z}(D_4) \)?

Orbits = average # fixed points.

An object is fixed by \( g \in G \) if its colors are constant across cycles.

Therefore:

\[ * \text{2-colored squares} \]
\[ = * \text{orbits} \]
\[ = \text{average} \# \text{fixed points} \]
\[ = \mathbb{Z}(D_4) \mid x_1 = x_2 = x_3 = x_4 = z \]
\[ = \frac{1}{4} \left( 2^4 + 3 \cdot 2^2 + 2 \cdot 2^2 + 2 \cdot 2 \right) \]
\[ = \frac{1}{4} \left( 16 + 12 + 16 + 4 \right) \]
\[ = 6. \]

But we can do more.

If \( g \) has a cycle of length \( k \), then (in a fixed object) we can have all \( k \) points (b)lue or all \( k \) points (r)ed.

So if we substitute

\[ x_1 = r + b \]
\[ x_2 = r^3 + b^3 \]
\[ x_3 = r^7 + b^7 \]
\[ x_4 = r^9 + b^9 \]

into \( \mathbb{Z}(D_4) \), we get

\[ \frac{1}{4} \left( \{r+b\}^4 + 3\{r+b\}^2 \right) + 2\{r+b\}\{r^3+b^3\} \]
\[ = r^4 + r^3 b + 2 r^2 b^3 + r b^3 + b^4. \]

This is the generating function for 2-colored squares "by color."

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**Counting graphs**

Graphs on labelled vertices \([n]\): \( \mathbb{Z}(2) \) - trivial

Graphs on labelled vertices \([n]\) that are connected:

- Complicated summation via Möbius inversion in the set partition lattice.

Graphs on unlabelled vertices?

Polya counting...

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**Unlabeled graphs on 4 vertices**

The group of symmetries here is \( S_4 \), but unlike previous examples, we want to know about the action of this group on edges.

<table>
<thead>
<tr>
<th>( S_4 )</th>
<th>action on edges</th>
<th>#</th>
<th>cycle index</th>
</tr>
</thead>
<tbody>
<tr>
<td>identity</td>
<td>identity</td>
<td>1</td>
<td>( x_4^6 )</td>
</tr>
<tr>
<td>2-cycle</td>
<td>2-cycles</td>
<td>6</td>
<td>( x_4^4 x_2^2 )</td>
</tr>
<tr>
<td>2-cycles</td>
<td>2-cycles</td>
<td>3</td>
<td>( x_4^2 x_2^2 )</td>
</tr>
<tr>
<td>3-cycle</td>
<td>3-cycles</td>
<td>8</td>
<td>( x_3^3 )</td>
</tr>
<tr>
<td>4-cycle</td>
<td>2-cycle &amp; 4-cycle</td>
<td>6</td>
<td>( x_4 x_2 )</td>
</tr>
</tbody>
</table>

The cycle index is therefore

\[ \frac{1}{24} \left( x_4^6 + 9 x_4^5 x_2^2 + 8 x_4^3 + 6 x_4 x_2 \right). \]
To find the generating function for unlabeled graphs on 4 vertices by the # of edges, we substitute:

\begin{align*}
X_1 &= 1 + t \\
X_2 &= 1 + t^2 \\
X_3 &= 1 + t^3 \\
X_4 &= 1 + t^4,
\end{align*}

and we get

\[1 + t + 2t^2 + 3t^3 + 2t^4 + t^5 + t^6.\]

Note that this polynomial is:

1) symmetric, and

2) unimodal.