Math 68. Algebraic Combinatorics.

Problem Set 1. Due on Friday, 10/7/2011

1. Prove that
\[
\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}
\]
for \( n \geq m \geq k \geq 0 \) by counting certain pairs of sets \((A, B)\) in two ways, and deduce that
\[
\sum_{k=0}^{m} \binom{n}{k} \binom{n-k}{m-k} = 2^m \binom{n}{m}.
\]

2. Give a combinatorial proof of the equality
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0,
\]
by describing a bijection between subsets of \([n]\) of odd size and subsets of \([n]\) of even size.

3. Let \([n] = \{1, 2, \ldots, n\}\).
   (a) Find the number of \(k\)-tuples \((S_1, S_2, \ldots, S_k)\) of subsets of \([n]\) such that \(S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k\).
   (b) Find the number of \(k\)-tuples \((S_1, S_2, \ldots, S_k)\) of subsets of \([n]\) such that \(S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset\).

4. A Delannoy path is a lattice path in \(\mathbb{Z}^2\) from \((0, 0)\) to \((m, n)\) using steps \((1, 0)\) (horizontal), \((0, 1)\) (vertical), and \((1, 1)\) (diagonal). The number of these paths is the Delannoy number \(D_{m,n}\). For example, \(D_{2,1} = 5\). Prove that
\[
D_{m,n} = \sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{m}.
\]
*Hint:* Classify the paths according to the number of diagonal steps.

5. Prove that the number of partitions of \(n\) into odd parts equals the number of partitions of \(n\) into distinct parts.

6. (a) In how many ways can we choose \(k\) points, no two consecutive, from a collection of \(n\) points arranged in a line?
   (b) What if the \(n\) points are arranged in a circle?

7. A set partition \(\pi\) of \([n]\) is a way to subdivide \([n]\) into nonempty blocks. A set partition is called noncrossing if it contains no two blocks \(B\) and \(B'\) such that \(i, k \in B\) and \(j, l \in B'\) for some \(i < j < k < l\). Show that the number of noncrossing partitions equals the Catalan number \(C_n = \frac{1}{n+1} \binom{2n}{n}\).