1. Prove that
\[
\frac{1}{1-z} = \prod_{j \geq 0} (1 + z^{2^j}).
\]

2. For fixed \(k\), give the exponential generating function for the number of surjective maps from \([n]\) onto \([k]\).

3. (a) Let \(b_n\) denote the number of (labeled) rooted trees on the vertex set \([n]\) whose leaves are colored either red or blue. Find an equation satisfied by the exponential generating function
\[
B(z) = \sum_{n \geq 0} b_n \frac{z^n}{n!} = 2z + 4\frac{z^2}{2!} + 24\frac{z^3}{3!} + \ldots
\]

(b) Use the Lagrange inversion formula to deduce that
\[
b_n = \sum_{k=0}^{n} \binom{n}{k} k^{n-1}.
\]

(c) * Give a direct combinatorial proof of (b).

4. Let \(M(n)\) be the set of all subsets of \([n]\), with the ordering \(A \leq B\) if the elements of \(A\) are \(a_1 > a_2 > \cdots > a_j\) and the elements of \(B\) are \(b_1 > b_2 > \cdots > b_k\), where \(j \leq k\) and \(a_i \leq b_i\) for \(1 \leq i \leq j\). (The empty set \(\emptyset\) is the bottom element of \(M(n)\).)

(a) Draw the Hasse diagrams (with vertices labeled by the subsets they represent) of \(M(1), M(2), M(3),\) and \(M(4)\).

(b) Show that \(M(n)\) is graded of rank \(\binom{n+1}{2}\). What is \(\text{rank}(\{a_1, \ldots, a_k\})\)?

(c) Define the rank-generating function of a graded poset \(P\) to be
\[
F(P, q) := \sum_{x \in P} q^{\text{rank}(x)}.
\]
Show that the rank-generating function of \(M(n)\) is given by
\[
F(M(n), q) = (1 + q)(1 + q^2) \cdots (1 + q^n).
\]

5. Let \(q\) be a prime power, and let \(V\) be an \(n\)-dimensional vector space over \(\mathbb{F}_q\). Let \(B_n(q)\) denote the poset of all subspaces of \(V\), ordered by inclusion. It’s easy to see that \(B_n(q)\) is graded of rank \(n\), the rank of a subspace of \(V\) being its dimension.
(a) Show that the number of elements of $B_n(q)$ of rank $k$ is given by the $q$-binomial coefficient
\[
\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1)\ldots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\ldots(q - 1)}.
\]

(One way to do this is to count in two ways the number of $k$-tuples $(v_1, \ldots, v_k)$ of linearly independent elements from $\mathbb{F}_q^n$: (1) first choose $v_1$, then $v_2$, etc., and (2) first choose the subspace $W$ spanned by $v_1, \ldots, v_k$, and then choose $v_1, v_2$, etc.)

(b) Show that $B_n(q)$ is rank-symmetric. (You can use (a).)

(c) Show that every element $x \in B_n(q)$ covers $\binom{k}{q} = 1 + q + \cdots + q^{k-1}$ elements and is covered by $\binom{n-k}{q} = 1 + q + \cdots + q^{n-k-1}$ elements.

(d) Define operators $U_i : \mathbb{R}B_n(q)_i \rightarrow \mathbb{R}B_n(q)_{i+1}$ and $D_i : \mathbb{R}B_n(q)_i \rightarrow \mathbb{R}B_n(q)_{i-1}$ by
\[
U_i(x) = \sum_{y \in B_n(q)_{i+1}, y > x} y, \quad D_i(x) = \sum_{z \in B_n(q)_{i-1}, z < x} z.
\]

Show that $D_{i+1}U_i - U_{i-1}D_i = (\lbrack n - i \rbrack_q - \lbrack i \rbrack_q)I_i$.

(e) Deduce that $B_n(q)$ is rank-unimodal and Sperner.

6. * Let $h_n$ be the number of ways to choose a permutation $\pi$ of $[n]$ and a subset $S$ of $[n]$ such that if $i \in S$, then $\pi(i) \not\in S$. Find an expression for the exponential generating function $\sum_{n \geq 0} h_n \frac{x^n}{n!}$. 
